Chapter 10

USING INDIFFERENCE POINTS IN ENGINEERING DECISIONS

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Proceedings of the
11th International Conference on Design Theory and Methodology
ASME, Paper Number DETC2000/DTM-14559, (September, 2000)

Abstract Multi-criteria decision support methods are common in engineering design. These methods typically rely on the specification of importance weights to accomplish trade-offs among competing objectives. Such methods can have difficulties, however: they may not be able to select all possible Pareto optima, and the direct specification of importance weights can be arbitrary and *ad hoc*. The inability to reach all Pareto optima is shown to be surmountable by the consideration of *trade-off strategy* as an additional parameter of a decision. The use of *indifference points* to select a best solution, as an alternative to direct specification of importance weights, is presented, and a simple truss design example is used to illustrate the concepts.

Keywords: multicriteria analysis, engineering design, design decision-making, aggregation functions, trade-offs, strategies

1. Introduction

Multi-criteria decision making is an important part of design. There are many methods, both informal and formal, that support such design decision making, such as Pugh charts [5], QFD [3], and the Analytic Hierarchy Process, or AHP [6]. These design decision methods share several key features. All rely on the aggregation of preferences to choose among designs, and most methods allow for the assignment of importance to individual attributes through the use of weights. These importance weights are meant to allow for meaningful comparison of many options when two or more attributes must be traded-off against each other. Among decision methods, weighted-sum aggregation of preferences is common, as is direct specification of importance weights.

Multi-criteria decision methods are related to multi-criteria optimization and the calculation of the Pareto frontier. Decision methods can be used to avoid unnecessary computation by optimizing directly to the most desirable configuration without calculating other Pareto points. It is implicit in the use of any decision method that the selection of its parameters, usually weights, enables the selection of the most desirable points.

Decision methods are important for decision support, and are crucial for semi-automated design, yet their underlying decision representations have rarely been examined or justified. Even if standard decision methods worked all the time, a formal investigation of the underlying mathematics of decision would still be warranted. It shall be demonstrated below that weighted-sum methods have serious drawbacks; in fact, any method that relies exclusively on importance weights to define a decision runs the risk of missing "optimal" options. A complete model of a decision requires an additional parameter to specify the level of compensation between criteria [7]. Also, the direct specification of importance weights is an *ad hoc* process, and the answers it produces may not always be reliable. Several relevant results are presented here:

- In addition to importance weights, the level of compensation between attributes is a parameter that defines a decision.
- For decision-making that conforms to the axioms of rational design [4], a parameterized family of functions (with compensation parameter s) was shown to span a complete range of degrees of compensation [8].
- The compensation parameter s increases with the level of compensation, which is demonstrated formally in two different ways. The compensation parameter, together with weights, defines a decision.
- By the use of these functions, a weight/strategy pair to select any Pareto optimal point can always be found.
- The ability to choose any Pareto point is *not* present when degree of compensation is pre-selected (as with the use of a weighted sum).

The concept of *indifference points* as a structured alternative to *ad hoc* specification of parameters is presented below, and its application illustrated by an example. The notion of level of compensation is a less intuitive concept than importance weighting, and a structured method is even more essential when both compensation and importance weights are considered. The relation of these results to multi-criteria optimization will be discussed below.

2. Example: simple truss structure

Consider the structure shown in Figure 10.1. This is a pin-jointed two-member bracket to support a load of one kilogram (1000 g) at a distance of

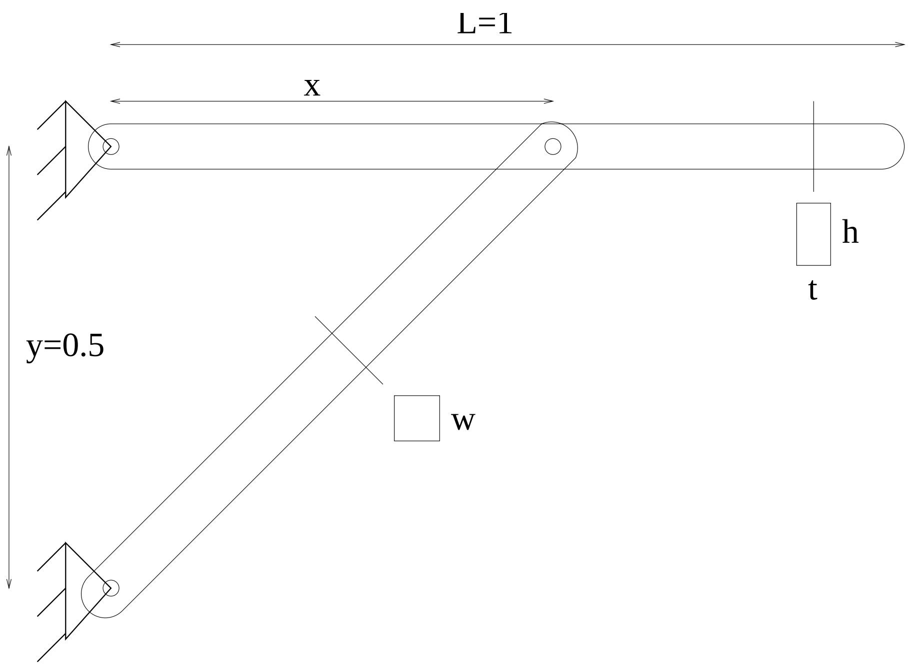

Figure 10.1 Example: a simple truss structure.

1 meter from a wall (L $=$ 1). The positions of the wall mounts are fixed, with the lower support one half meter below the upper support ($y = 0.5$). Both members are made of aluminum (6061-T6). The designer controls four design variables:

$x \in [0.1\text{ m}, 0.9\text{ m}]$	distance from wall to pin
$t \in [5\text{ mm}, 20\text{ mm}]$	thickness of bending member
$h \in [5\text{ mm}, 20\text{ mm}]$	height of bending member
$w \in [5\text{ mm}, 20\text{ mm}]$	width of (square) compression member

For this example, the performance measures to consider are the mass (M) of the structure, and the safety factor (S). The example is simple enough that both can be expressed analytically, but let us start by treating the performance calculation as a black box. The details of the calculations are presented in the Appendix. For purposes of this paper, the design problem is to minimize the mass of the structure while maximizing the factor of safety.

Further suppose that no additional advantage is gained from factors of safety above ten. Also, designs with safety factors below one should not be considered. Using optimization or other means, it can be determined that the minimum mass achievable with a factor of safety of one is 123 grams, while the

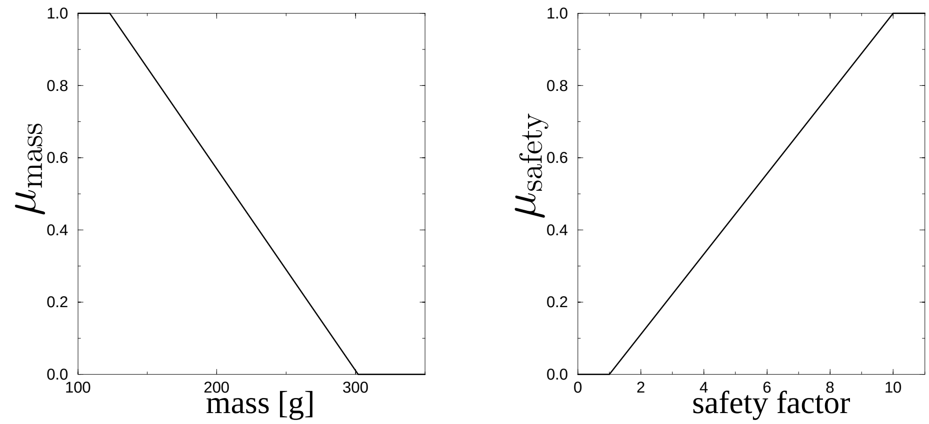

Figure 10.2 Preferences for mass and safety factor

minimum mass achievable with a factor of safety of ten is 302 grams. The best designs will be trade-offs between the safety factor and the mass.

Assuming that both 123 g (the lowest possible mass for the acceptable range of safety factors) and 302 g (the mass corresponding to the highest safety factor) are acceptable, it is common to normalize the performance measures. Here we follow the approach of the Method of Imprecision, or M_oI [9, 7], and specify preferences on the performance measures, which incidentally normalizes the performances to the interval $[0, 1]$. The results presented apply to any normalization scheme, or to no normalization at all. Let the preferences for mass and safety be as follows:

$$\mu_{\text{mass}}(\text{M}) = \frac{302 - \text{M}}{179}$$
$$\mu_{\text{safety}}(\text{S}) = \frac{\text{S} - 1}{9}$$

so that $\mu_{\text{mass}}(123) = 1$, $\mu_{\text{mass}}(302) = 0$, $\mu_{\text{safety}}(1) = 0$, and $\mu_{\text{safety}}(10) = 1$, as shown in Figure 10.2. Note that these simple linear preferences are chosen for convenience; all the results presented here hold for more complicated preferences as well.

Weighted sum

As was discussed above, a common way to select a best design is to assign importance weights to the two criteria, and then use a weighted sum to aggregate preferences; the best designs will have the highest overall preference. Let the importance weights assigned to mass (ω_1) and safety (ω_2) be normalized

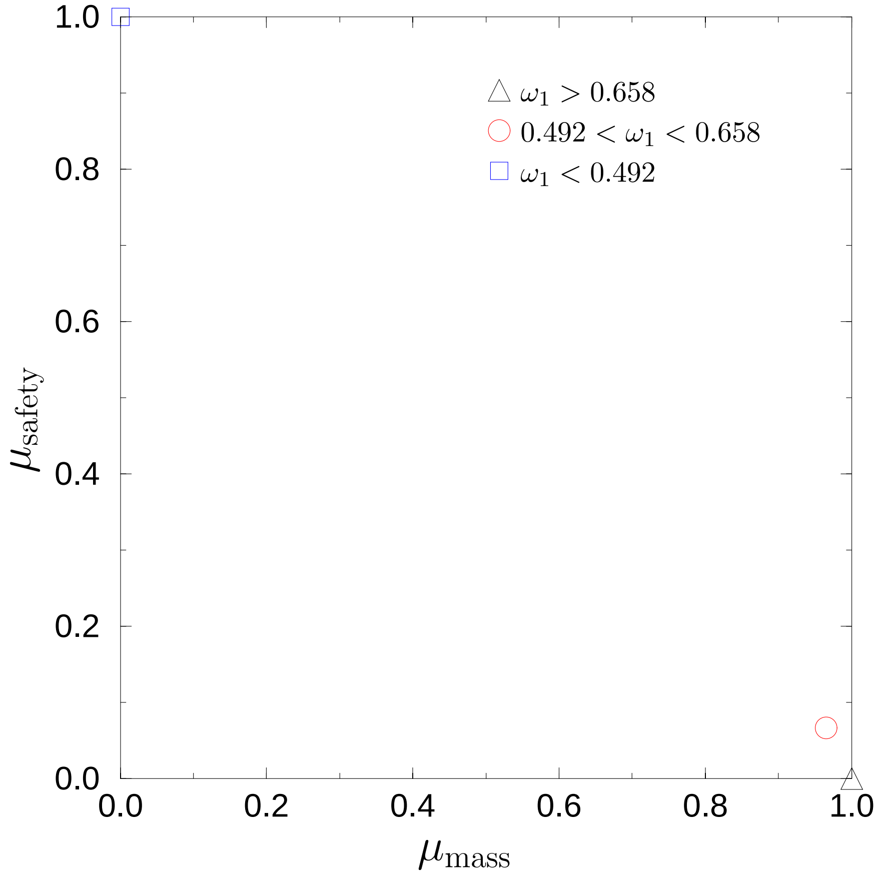

Figure 10.3 Three "best" points found using weighted sum exploration

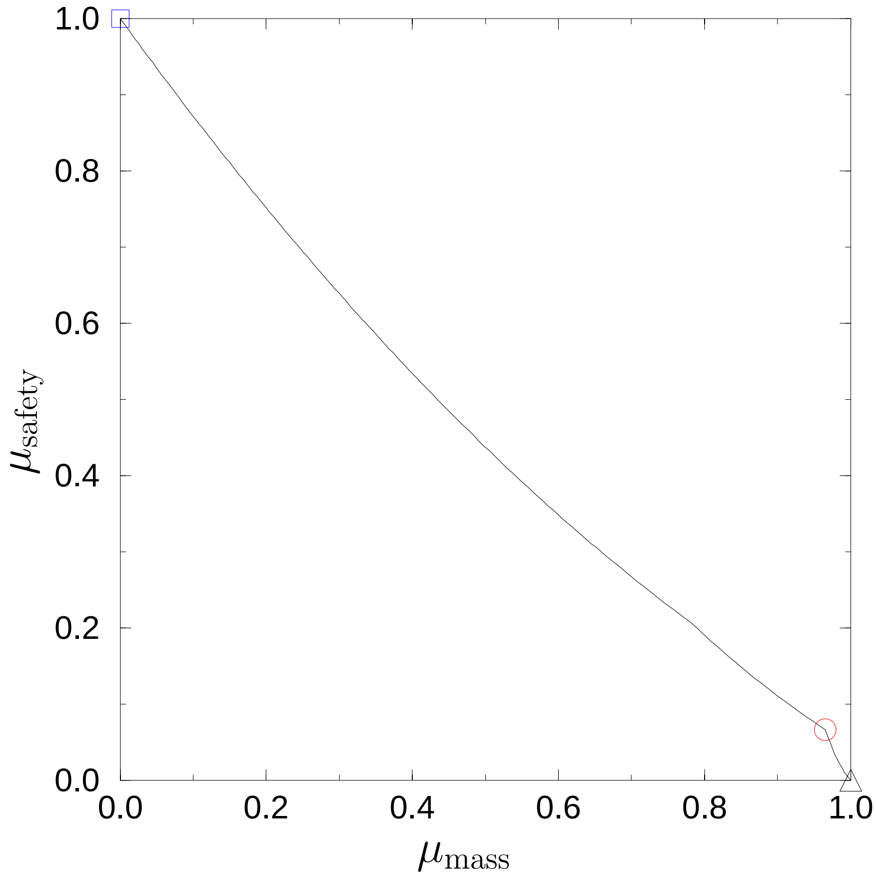

Figure 10.4 Pareto frontier with three "best" points.

so that their sum is one. Employing this approach, there are only three possible "best" points, summarized in the table below:

ω_1	mass [g]	safety factor	x	t	h	w
$\omega_1 > 0.658$	123	1	0.71	5	5	5
$0.492 < \omega_1 < 0.658$	129	1.6	0.82	5	5	5
$\omega_1 < 0.492$	302	10	0.9	5	9.34	8.16

According to this weighted sum aggregation, all other possible points are worse when both mass and safety factor are considered. These points are shown on the graph in Figure 10.3.

The three "best" points shown in Figure 10.3 do not represent the entire range of reasonable trade-offs between the two performance measures mass and safety. To make the notion of a “reasonable” decision more precise, we use the idea of Pareto optimality:

Definition 1 *The alternative* A dominates *the alternative* B *if* A *performs no worse than* B *on all attributes, and better than* B *on at least one attribute. In this case, regardless of the weights or the strategy, it is always better to choose* A *over* B. *A feasible solution is* undominated *(or* Pareto optimal*) if there is no other feasible solution which dominates it.*

Figure 10.4 shows a plot of all the feasible Pareto optimal points, normalized with respect to preference, with the three points from Figure 10.3 retained. (The calculation of the Pareto frontier is detailed in the Appendix). Examining the entire Pareto frontier, we see that it is made up of two concave sections. The weighted sum approach fails to identify any Pareto points on the concave sections of the frontier, though it is quite possible that one of those points represents the most desirable compromise. Some difficulties of the weighted sum have been discussed previously in an optimization context by [2] and [1], among others.

Clearly, a weighted sum approach to multi-criteria decision making is problematic if it cannot identify all possible best solutions. In the example presented here, the performance calculations are all analytic expressions which are easily evaluated, and the Pareto frontier is thus easily discovered. In such simple cases, a designer can choose to informally explore regions of the performance space that the formal decision model does not identify. When the design is more complicated, perhaps because evaluation is more costly (say, each point is a finite element calculation rather than an analytic expression), or because there are more than two competing objectives, informal exploration becomes much more difficult, and designers may rely more on automated techniques such as optimization. In these more complicated design situations, it is particularly important that design decision methods provide reliable guidance. If the preference aggregation is valid, it is not necessary to compute the entire Pareto frontier.

In the example presented above, it is clear that the choice of a point on the Pareto frontier depends on the trade-off between safety and mass. As is suggested by the problems exhibited by the weighted sum, a formal model of a design decision is more complex than a simple matter of choosing importance weights.

3. Compensation strategies: how to consider <u>all</u> designs

In the preceding section it was seen that a weighted sum cannot always identify all Pareto points for a design. This is one instance of a more general result about the aggregation of preference. All existing support methods for multi-criteria decision making ultimately rely on the aggregation of disparate preferences with *aggregation functions*. [4] presented axioms that an aggregation function must obey in order to be appropriate for rational design decision making. [8] showed that the operators that satisfy these axioms are a restricted set of weighted means, and that, in particular, there is a family of aggregation operators $\mathcal{P}_s$ that spans an entire range of possible operators between $\min$ and $\max$, given by:

$$\mathcal{P}_s(\alpha_1, \alpha_2; \omega_1, \omega_2) = \left(\frac{\omega_1 {\alpha_1}^s + \omega_2 {\alpha_2}^s}{\omega_1 + \omega_2} \right)^{\frac{1}{s}}$$

Here, the values α_1, α_2 are individual preference values to be aggregated. The values α_i are the result of applying preferences μ_i to performances x_i: $\alpha_i = \mu_i(x_i)$, or more generally, $\alpha = \mu(\vec{x})$. The parameter s can be interpreted as a measure of the *level of compensation*, or *trade-off*, and is sometimes referred to as the *trade-off strategy*. Higher values of s indicate a greater willingness to allow high preference for one criterion to compensate for lower values of another. The parameters ω_1 and ω_2 are importance weights, both assumed to be positive without loss of generality, and as they may be normalized, the ratio $\omega = \frac{\omega_2}{\omega_1}$ is sufficient to characterize the relative importance of two attributes. The definition is for two attributes, but can be extended to more than two. It is readily shown [8] that

$$\begin{aligned}
\mathcal{P}_{-\infty} &= \lim_{s \to -\infty} \mathcal{P}_s &&= \min \\
\mathcal{P}_0 &= \lim_{s \to 0} \mathcal{P}_s &&= \text{geometric mean } (\alpha_1^{\omega_1} \alpha_2^{\omega_2})^{\frac{1}{\omega_1 + \omega_2}} \\
\mathcal{P}_1 &= \lim_{s \to 1} \mathcal{P}_s &&= \text{arithmetic mean } \frac{\omega_1 \alpha_1 + \omega_2 \alpha_2}{\omega_1 + \omega_2} \\
\mathcal{P}_{\infty} &= \lim_{s \to +\infty} \mathcal{P}_s &&= \max
\end{aligned}$$

Thus the common weighted sum is one instance of this family of design-appropriate aggregation functions, with the compensation parameter s equal to 1.

Several results about this family of aggregation functions can be proven [7], including:

- For any Pareto optimal point in a given set, there is always a choice of a weight ratio ω and a trade-off strategy s that selects that point as the most preferred.

- For any fixed strategy s, there are Pareto sets in which some Pareto points can *never* be selected by any choice of weights ω. This is what occurs in the truss example above, where $s = 1$ cannot select all Pareto points. This is related to the well-known result that non-convex portions of a Pareto surface are unreachable by weighted-sum minimization.

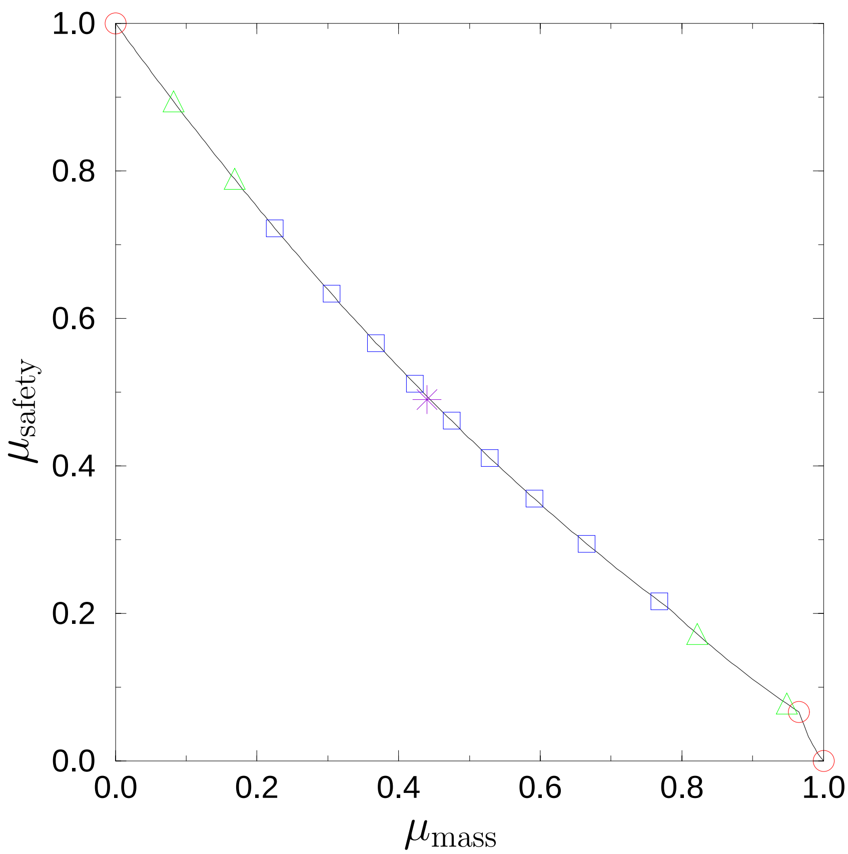

Figure 10.5 Normalized Pareto frontier

The second result does not say that for every strategy s, every Pareto set has some Pareto points that are unreachable. In the truss example, for instance, $s = -1$ can select every Pareto point if the correct weights are chosen. This can be seen in Figure 10.5. Here, the three "optimal" points found earlier by the weighted sum method are circled. Setting $s = -1$ and varying the weights allows for a more varied range of "best" designs:

ω_1	mass [g]	safety	x	h	w
0.1	262	7.5	0.9	8.09	7.59
0.2	248	6.7	0.9	7.65	7.38
0.3	236	6.1	0.9	7.30	7.21
0.4	226	5.6	0.9	6.99	7.06
0.5	217	5.15	0.9	6.70	6.91
0.6	207	4.7	0.9	6.41	6.76
0.7	196	4.2	0.9	6.05	6.57
0.8	183	3.65	0.9	5,64	6.34
0.9	164	2.95	0.9	5.07	6.01

These points are shown as squares on the graph in Figure 10.5.

By allowing the weight assigned to one attribute to be much larger than the weight assigned to the other, points much closer to the extremes of the Pareto frontier can be reached with $s = -1$:

ω_1	mass [g]	safety	x	h	w
0.01	288	9.05	0.9	8.89	7.96
0.05	272	8.1	0.9	8.41	7.74
0.95	155	2.55	0.89	5	5.77
0.99	132	1.7	0.83	5	5.10

These points are shown as triangles on the graph in Figure 10.5.

4. Using indifference points to determine strategies and weights

As was mentioned above, the direct specification of importance weights is an *ad hoc* process. If a designer says that safety is twice as important as mass, how are we to know that aggregation with any strategy will choose the best alternatives? The difficulty is only compounded by the consideration of trade-off strategies. In this section a technique is presented for determining the correct parameter pair of compensation strategy s and weight ratio $\frac{\omega_2}{\omega_1}$ for a particular decision. Rather than direct specification, the technique relies on the use of *indifference points* to establish the appropriate parameters.

Two points are considered *indifferent* if they have the same preference; it is not necessary that the numerical preferences be known. When a single individual has complete decision-making authority, strategies and weights can be considered simultaneously, and their values can be calculated from indifference points. The procedure is as follows:

1. Determine preferences $\alpha_1 = \mu_1(x_i)$ and $\alpha_2 = \mu_2(x_i)$ such that

$$\mathcal{P}_s(\alpha_1, 1; \omega_1, \omega_2) = \mathcal{P}_s(1, \alpha_2; \omega_1, \omega_2) = 0.5$$

(The values for s, ω_1, and ω_2 are to be determined.) In other words, at which value α_1 is there indifference between a design with a preference equal to α_1 for the first performance attribute and a preference values of 1 on the second attribute, and a design that achieves preferences of 0.5 on both attributes (and thus, by idempotency [4], has a combined preference of 0.5)? A similar question is asked for α_2. Sometimes it is easier to ask for values of x_i and calculate α_i; sometimes it is easier to seek α_i directly. Either approach to determining the indifference points is acceptable.

To see how the selection of indifference points might work on the truss example, start with the preference values from Section 2.:

μ	mass [g]	safety
0	302	1
0.5	214	5.5
1	126	10

The reference design is thus the one that yields a mass of 214 g and a safety factor of 5.5. It is important to note that the reference design does not need to be physically realizable. In the truss example, there is no feasible design with preferences $(0.5, 0.5)$. To determine the value of α_1, ask "If we start from the reference design and increase the safety factor to 10, how much can the mass increase so that the new design has the same overall preference as the reference design?" The answer could be, say, that the mass can increase to 260 g; since $\mu_1(260) = 0.29$, $\alpha_1 = 0.29$, indicating that there is indifference between the two points with performances $(x_1, x_2) = (260, 10)$ (preferences $(\alpha_1, \alpha_2) = (0.29, 1)$) and $(214, 5.5)$ (preferences $(0.5, 0.5)$).

The corresponding question is asked for α_2: "Comparing the reference design to one where the mass is 126 g, what is the safety factor that achieves indifference with the reference design (214,5.5)?" If the answer is that a safety factor of 3 ($\mu_2(3) = 0.22$) together with a mass of 214 g is indifferent to a safety factor of 5.5 together with a mass of 126 g, then $\alpha_2 = 0.22$.

If the person providing the indifference points is comfortable thinking in terms of preferences between 0 and 1, then it is not necessary to refer to performance values. Instead the question can be asked directly: "If we start with a reference design where preference for both mass and safety is 0.5, and we increase the preference for mass to 1, how low a preference for safety achieves indifference with the reference design?" This latter form of questioning is particularly useful when several attributes are combined hierarchically and thus groups of attributes must be compared. It is always possible, even in hierarchical aggregation, to specify particular values of all performance attributes in order to specify indifference points.

2. Let $\omega = \frac{\omega_2}{\omega_1}$.
3. If $\alpha_1 = \alpha_2$, then $\omega = 1$:
   (a) If $\alpha_1 = 0.5$, then $s = -\infty$.
   (b) If $\alpha_1 = 0.25$, then $s = 0$.
   (c) If $\alpha_1 > 0.25$, then $s \in (-\infty, 0)$. Solve ${\alpha_1}^s + 1 = 2(0.5)^s$ numerically.
   (d) If $\alpha_1 < 0.25$, then $s \in (0, \infty)$. Solve ${\alpha_1}^s + 1 = 2(0.5)^s$ numerically.
4. If $\alpha_1 \neq \alpha_2$, then $\omega \neq 1$. Note that if $s = 0$:

$$ {\alpha_1}^m = 0.5 = {\alpha_2}^{1-m} \Rightarrow {\alpha_2}^{1-\log_{\alpha_1} 0.5} = 0.5 $$

   Thus:

   (a) If ${\alpha_2}^{1-\log_{\alpha_1} 0.5} = 0.5$, then $s = 0$, and $b = \frac{1-\log_{\alpha_1} 0.5}{\log_{\alpha_1} 0.5}$
   (b) If ${\alpha_2}^{1-\log_{\alpha_1} 0.5} > 0.5$, then $s < 0$.
   If ${\alpha_2}^{1-\log_{\alpha_1} 0.5} < 0.5$, then $s > 0$.
   Solve numerically for s from

$$ \left(\frac{1 + \omega {\alpha_2}^s}{1 + b} \right)^{\frac{1}{s}} = \left(\frac{{\alpha_1}^s + \omega}{1 + \omega} \right)^{\frac{1}{s}} = 0.5 $$

   which reduces to

$$ \left({\alpha_1}^s - 0.5^s\right)\left({\alpha_2}^s - 0.5^s\right) = \left(1 - 0.5^s\right)^2 $$

   Once this is solved numerically for s, then ω can also be determined.

Applying steps 2–4 to the indifference values $\alpha_1 = 0.29$ and $\alpha_2 = 0.22$ determined above, the parameters that determine the aggregation are found to be:

$$ s^* = -0.5 \quad \omega^* = 1.23, \text{ or } (\omega_1, \omega_2) = (0.45, 0.55) $$

The best point on the Pareto frontier for that trade-off is:

mass [g]	safety	α_1	α_2	x	t	h	w
223	5.45	0.44	0.49	0.9	5	6.90	7.01

Note that this best point depends only on the preferences for safety and mass, and on the answers given above to the two indifference questions. Also note that this point, which is shown as a star on the graph in Figure 10.5, is intuitively appealing as an overall optimum, while the points provided by the weighted sum method are not.

It should be noted that if either α_1 or α_2 is close to 0, then the (s, ω) pair is quite sensitive to small differences in α_1 and α_2. In these cases, it might be preferable to elicit other indifference points to determine s and ω. In the procedure described above, points that are equivalent to $(0.5, 0.5)$ are chosen; the procedure can easily be modified to consider indifference to some other reference point. Indeed, if the procedure is applied more than once with different reference points, the redundant information serves as a check on the accuracy of the specification.

5. Conclusion

Weighted sum aggregation with importance weights is common to many methods for engineering design decision making. Two important difficulties with these methods are the inability of weighted-sum methods to select all Pareto points, and the arbitrary nature of direct assignment of importance weights.

It is shown here that a complete model of an engineering decision depends not only on the importance weights, but also on the level of compensation, or trade-off strategy. A weighted sum, or any other pre-determined aggregation procedure, is overly and inappropriately constraining. The appropriate strategy (or degree of compensation among attributes) is situation-dependent, and a "one-size-fits-all" decision method that dictates an aggregation method can lead to incorrect results. An easily computed family of preference aggregation functions is completely determined by two parameters that represent the trade-off strategy (degree of compensation) and the importance weighting.

The choice of strategy and importance weights, or even the choice of importance weights alone, can be *ad hoc* and arbitrary if accomplished by direct specification. A simple procedure is presented, along with straightforward calculations, to establish the proper importance weights and degree of compensation to reach rational engineering decisions.

The results in this paper regarding decision methods are related to previously known results about multi-criteria optimization [2, 1]. The preference aggregation operators presented here could be used to explore a non-convex Pareto frontier. From an optimization point of view, once an aggregated preference function (or utility function) is determined, locating the optimum is straightforward. If some point lies on a non-convex region of a Pareto frontier, and a utility function is constructed using a weighted sum, then that point is an inferior point. From a design decision point of view, however, it is appropriate to question the choice of a weighted sum to aggregate the preferences. The method presented in this paper, unlike a weighted sum, has the great advantage that it does not *a priori* exclude any Pareto points from consideration. Thus, if the aggregated preference is optimized, the selected "optimum" is actually what the designer desires, and is not artificially constrained by the geometry of the design and performance spaces.

Acknowledgements

This material is based upon work supported, in part, by the National Science Foundation under NSF Grant Number DMI-9813121. Any opinions, findings, conclusions, or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the sponsors.

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A Appendix: Details of the Truss Example

Here are the details for the example of the bracket shown in Figure 10.1.

Material properties and performance measures

The material is aluminum (6061-T6), which has the following relevant material properties:

Young's modulus (E)	$69 \cdot 10^9$ Pa
Density (ρ)	$2660\ \mathrm{kg/m^3}$
Yield stress (σ)	$275 \cdot 10^6$ Pa

There are four design variables:

$x \in [0.1\ \mathrm{m}, 0.9\ \mathrm{m}]$	distance from wall to pin
$t \in [5\ \mathrm{mm}, 20\ \mathrm{mm}]$	thickness of bending member
$h \in [5\ \mathrm{mm}, 20\ \mathrm{mm}]$	height of bending member
$w \in [5\ \mathrm{mm}, 20\ \mathrm{mm}]$	width of (square) compression member

The first performance measure is total mass (in kilograms, here, so the load P below is equal to 1):

$$\mathrm{M} = \rho \left(htL + w^2 \sqrt{x^2 + y^2} \right) = 2660 \left(ht + w^2 \sqrt{x^2 + 0.25} \right)$$

The safety factor S has two components, the safety factor for the bending member S_b , and the safety factor for the compression member S_c. Since the yield stress in the bending member is σ, and the maximum stress in the bending member is $\frac{12P(L-x)}{th^2}$, the factor of safety in bending is the ratio:

$$\mathrm{S}_b = \frac{\sigma t h^2}{12P(L-x)} = \frac{\sigma t h^2}{120(1-x)}$$

Similarly, using the Euler buckling load, the safety factor in the compression member is:

$$\mathrm{S}_c = \frac{\pi^2 \mathrm{E} x y w^4}{12PL(x^2+y^2)^{1.5}} = \frac{\pi^2 \mathrm{E} x y w^4}{120(x^2+0.25)^{1.5}}$$

The safety factor for the entire design is defined to be the minimum of the two:

$$\mathrm{S} = \min \left(\frac{\sigma t h^2}{120(1-x)}, \frac{\pi^2 \mathrm{E} x y w^4}{120(x^2+0.25)^{1.5}} \right)$$

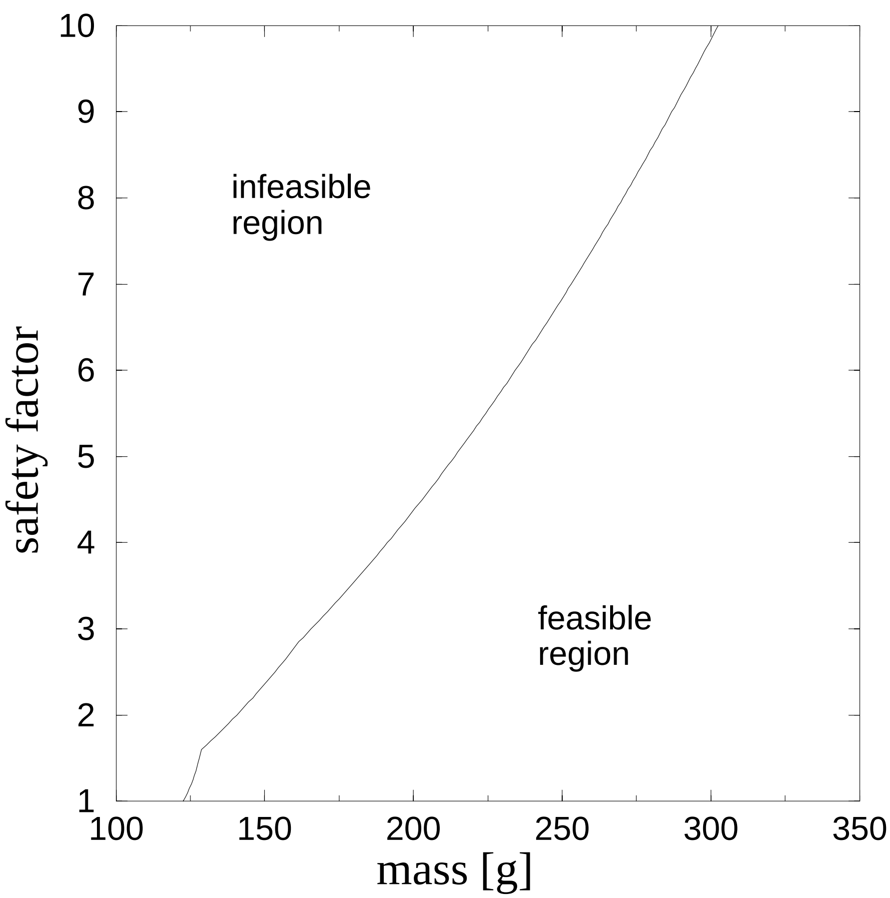

Figure A.1 Pareto frontier of best performances

Calculation of the Pareto frontier

The design problem is to minimize the mass while maximizing the factor of safety; both are analytic expressions. First, note that mass is linear in both t and h, while the factor of safety in bending is linear in t but quadratic in h. Thus, as long as no other design variables reach the maximum acceptable dimensions, it will always be preferable to increase h rather than t. Setting $t = t_{\rm min} = 5\ {\rm mm}$ reduces the problem to the three design variables x, h, and w. Both h and w can be expressed as functions of x and a safety factor, and thus finding the minimum possible mass for a given safety factor requires solving a rational equation in x. These solutions yield a Pareto frontier of designs, which is shown in Figure A.1.

The values of x, h, and w which generate these optimal designs are included in Figure A.2. It can be seen from Figure A.2 that at each Pareto point, at least one domain constraint is active: in particular, for low mass, h takes its minimum acceptable value of 5 mm, while for higher mass, x takes its maximum acceptable value of 0.9 m. Nevertheless, along most of the Pareto frontier two design variables are changing as the frontier is traversed.

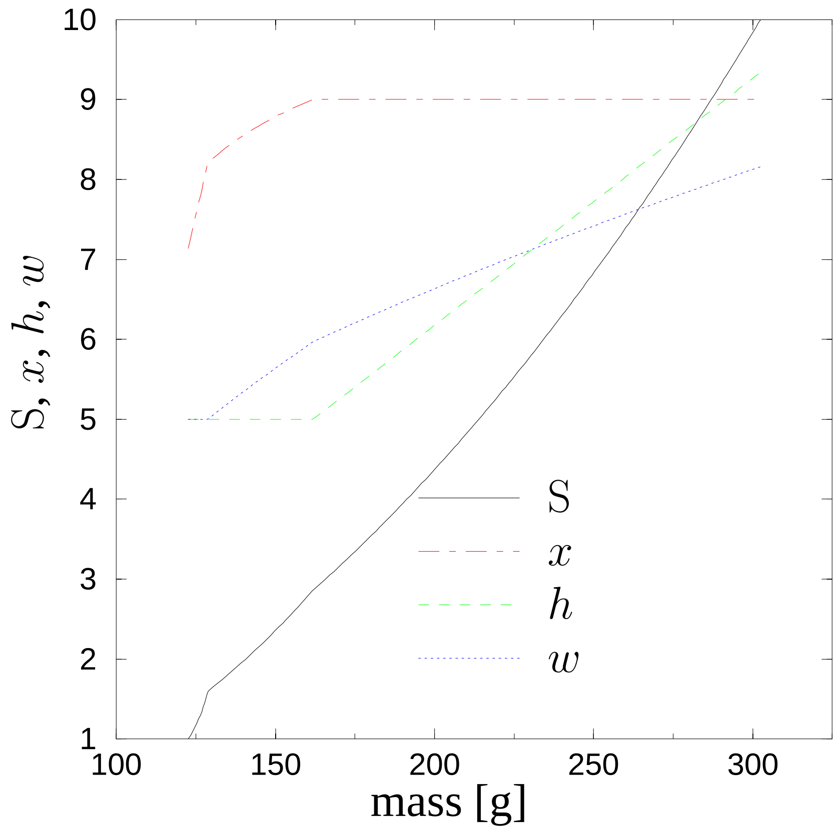

Figure A.2 Pareto frontier with design variable values