Chapter 6

AGGREGATION FUNCTIONS FOR ENGINEERING DESIGN TRADE-OFFS

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Abstract The choice of an aggregation function is a common problem in Multi Attribute Decision Making (MADM) systems. The *Method of Imprecision* (M_oI) is a formal theory for the manipulation of preliminary design information that represents preferences among design alternatives with the mathematics of fuzzy sets. The M_oI formulates the preliminary design problem as a MADM problem. To date, two aggregation functions have been developed for the M_oI, one representing a compensating strategy and one a non-compensating strategy. Much of the prior fuzzy sets research on aggregation functions has been inappropriate for application to engineering design. In this paper, the selection of an aggregation function for MADM schemes is discussed within the context of the M_oI. The general restrictions on design-appropriate aggregation functions are outlined, and a family of functions, modeling a range of trade-off strategies, is presented. The results are illustrated with an example.

Keywords: Fuzzy design; Engineering; Aggregation functions; Averaging operators; Decision making; Multicriteria Analysis

Introduction

Preliminary design decisions are among the most important in engineering. It is when the details of a design are unknown and the design description is still *imprecise*, that the most costly and important decisions are made [19]. A rigorous treatment of design decision-making is necessary for the construction of computer tools to aid the designer in that decision-making. One such tool is the *Method of Imprecision* (M_oI) [20], a formal method for preliminary design decision-making that uses the mathematics of fuzzy sets to represent and manipulate imprecise design information. The M_oI casts the design decision-making problem as a Multi Attribute Decision Making (MADM) or Multi Objective Decision Making (MODM) problem [4, 24].

The choice of an aggregation function is a crucial part of any MADM or MODM scheme. The M_oI has attempted to approach this choice axiomatically. The axioms of the M_oI restrict the choice of candidate aggregation functions so as to reflect the natural structure of the design decision-making process [10, 11]. Within the axiomatic framework of the M_oI, a designer can combine preliminary design information using different trade-off strategies. In particular, two aggregation functions have been used for multi-attribute decision making, one which provides a compensation between the criteria, and one which does not compensate.

Many different aggregation functions for fuzzy sets have been proposed and studied. Much fuzzy set research has focused on *t-norms* and *t-conorms* [5]. T-norms are bounded above by the *min* function, and are the appropriate model for extensions of the logical AND to fuzzy sets, while t-conorms are bounded below by the *max* function and are an extension of the logical OR. Both classes of functions have been studied in detail, and research is on-going (*e.g.*, [14]). MADM schemes have applied a wide range of t-norms and other operators to decision problems. While averaging operators, which fall between *min* and *max*, have been acknowledged for some time [5, 22, 23], comparatively little study has been devoted to these connectives. Averaging operators are not appropriate for binary logic, but they are well suited to engineering design decisions: indeed, the axioms of the M_oI require its aggregation functions to fall between *min* and *max* [10]. This paper will offer a more thorough treatment of averaging operators than has been presented previously.

This paper discusses the problem of choosing an aggregation function for multiple criteria decision making. The results are presented within the context of the M_oI, but they have general applicability in MADM and MODM approaches. The axioms of the M_oI and the reasons for their use in modeling engineering design decisions will be presented. In the context of these axioms, potential aggregation functions will be discussed. The M_oI presently uses two different aggregation functions to model two different design trade-off strategies. This paper will explore the range of possible aggregation functions suitable for such decision-making. A parameterized family of functions that satisfies all the axioms for design and that models a continuum of strategies between the two existing strategies of the M_oI will be presented. The possibility of suitable aggregation functions outside this range will be discussed, and an example of an application will be given.

The organization of the paper is as follows. The first section presents the issue of uncertainty in engineering design and introduces the M_oI. The second section discusses the axioms of the M_oI in more detail. The third section covers some of the prior research on fuzzy MADM systems. In the fourth section, a parameterized family of aggregation functions is presented. The fifth section contains some philosophical remarks on the implications of this paper. Finally, the sixth section illustrates the application of the result with an example.

1. Uncertainty in Engineering Design

In preliminary engineering design, the final values of *design variables* are uncertain. As this uncertainty is not probabilistic, but will be resolved by further refinement and specification later in the design process, it is appropriately modeled using the mathematics of fuzzy sets [20, 21]. The Method of Imprecision is a formal system for the representation and manipulation of such imprecise design data. When a design is finalized, standard analysis tools (*e.g.*, finite element analysis, *etc.*) serve to calculate the performance of a design. These standard analysis tools do not operate on the imprecise information that is available during preliminary design, and designers rely on experience and intuition in this early stage. In the M_oI, designers represent their preferences for different values of a design variable using fuzzy sets: each value of a design variable is assigned a preference between zero (totally unacceptable) and one (completely acceptable). Design variables can take on discrete, continuous, or linguistic values. The designer's preferences create a fuzzy set that could be called "Values of design variable d preferred by the designer"; the membership in this set of a particular value of d is the designer's preference for that value. In this way, the designers' judgment and experience are formally incorporated into the preliminary problem. The designers' preferences, expressed on the design variables, are mapped to the performance space where they become preferences on the *performance variables* [8]. Preferences are also specified directly on performance variables by designers or others, and represent the functional requirements for the design. Whether calculated or expressed directly, these performance variables are likewise represented with fuzzy sets. Since a design's performance is usually described by several different performance variables, reconciliation of competing aspects of a design's performance forms an important part of the M_oI.

The general problem is thus a Multiple Attribute Decision Making (MADM) problem: the designer is to choose the highest performing design configuration from an available design space, when each design is to be judged by several, perhaps competing, performance criteria (variables). These performance variables may have different levels of importance, or *weights*. The MADM problem is one of the aggregation of weighted fuzzy sets. The goal is to choose an *aggregation function* that properly models the concerns of a designer evaluating a design.

A few points of notation: preferences are denoted by μ, indexed as μ_i, and their attendant weights are denoted by ω_i. The overall preference μ_o is calculated using an aggregation function $\mathcal{P}$:

$$\mu_o = \mathcal{P}((\mu_1, \omega_1), \cdots, (\mu_n, \omega_n))$$

If it is convenient to ignore the weights, as in the common case of equal weights for all objectives, the function can be written simply

$$\mu_o = \mathcal{P}(\mu_1, \cdots, \mu_n)$$

The MoI has previously used two different aggregation functions to model two different situations in decision-making in design. When the overall preference for the performance of a design is limited by the attribute with the lowest performance, the decision-making problem is said to be *non-compensating*, and the aggregation function used is the simple minimum. In this case, weights are immaterial, and are not included in the calculations. When good performance on one attribute is perceived to partially compensate for lower performance on another, the problem is called *compensating*, and the geometric weighted mean or product of powers has been used.[1] Which aggregation functions are appropriate for design decision-making is the subject of this paper.

2. The Axioms of the MoI

In any MADM system, it is desirable that the aggregation functions used be justifiable models for decision-making behavior. The choice of an aggregation function may be justified in several ways. Empirical tests, such as those conducted by other researchers [15] can help determine which aggregation functions best model human decision-making in various contexts. (The study mentioned supports the aggregation functions used by the MoI.) Computational simplicity is often used as a basis for the choice of an aggregation function, however inappropriate that may be. The development of the MoI has been to appeal to intuitive notions of rational human behavior [16], and to formalize this rationality in a set of axioms that the aggregation functions must follow. The axioms of the MoI (see Table 6.1) [10] are a formal description of restrictions on any aggregation function for (rational) engineering design.

The axioms of monotonicity, commutativity and continuity are common to many multi-attribute decision making schemes, and they are uncontroversial. Also uncontroversial is the idea that an attribute with zero weight should contribute nothing to the overall performance. Self-scaling weights are convenient

[1] Both of these aggregations are analogous to Pareto-optimal solutions in game theory [9, 18], and the product of powers is analogous to a Nash solution. However, neither correspondence is mathematically precise, since preferences are not equivalent to utilities.

Monotonicity: $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n)) \leq \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu'_n,\omega_n))$ for $\mu_n \leq \mu'_n$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n)) \leq \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega'_n))$ for $\omega_n \leq \omega'_n;\ \ \mu_i < \mu_n\ \ \forall i < n$
Commutativity: $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_i,\omega_i),\cdots,(\mu_j,\omega_j),\cdots,(\mu_n,\omega_n)) =$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_j,\omega_j),\cdots,(\mu_i,\omega_i),\cdots,(\mu_n,\omega_n))\ \forall i,j$
Continuity: $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega_k),\cdots,(\mu_n,\omega_n)) =$ $\lim_{\mu'_k\to\mu_k}\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu'_k,\omega_k),\cdots,(\mu_n,\omega_n)) \quad \forall k$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega_k),\cdots,(\mu_n,\omega_n)) =$ $\lim_{\omega'_k\to\omega_k}\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega'_k),\cdots,(\mu_n,\omega_n)) \quad \forall k$
Idempotency: $\mathcal{P}((\mu,\omega_1),\cdots,(\mu,\omega_n)) = \mu$ for $\omega_1,\cdots,\omega_n \geq 0;\ \omega_1+\cdots+\omega_n > 0$
Annihilation: $\mathcal{P}((\mu_1,\omega_1),\cdots,(0,\omega),\cdots,(\mu_n,\omega_n)) = 0$ for $\omega \neq 0$
Self-scaling weights: $\mathcal{P}((\mu_1,\omega_1 t),\cdots,(\mu_n,\omega_n t)) = \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n))$ $\forall\ \omega_1,\cdots,\omega_n \geq 0;\ \omega_1+\cdots+\omega_n, t > 0$
Zero weights: $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,0),\cdots,(\mu_n,\omega_n))$ $= \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_{k-1},\omega_{k-1}),(\mu_{k+1},\omega_{k+1}),\cdots,(\mu_n,\omega_n))$

Table 6.1 Axioms of the M$_o$I

for the hierarchical combination of an arbitrary number of attributes. Two of the axioms, however, idempotency and annihilation, are fundamental to the $\mathrm{M_oI}$ and are particularly relevant here.

The idempotency axiom appeals to a notion of rational behavior. It states that if several variables with identical preferences are combined, the overall preference must be the same as the (identical) preferences on the individual variables. Idempotency reflects the constraint that the overall preference for a design should never exceed the preference of the highest-ranked attribute, nor fall below the preference of the lowest-ranked attribute. Idempotency and monotonicity together lead to the requirement that $\mathit{min} \leq \mathcal{P} \leq \mathit{max}$.

The annihilation axiom is specific to engineering design, and others have argued its validity [3, 12, 17]. It states that if the preference for any one attribute of the design sinks to zero (unacceptable) then the overall preference for the design is zero. For example, given a fixed material, the tensile strength limit cannot be exceeded no matter the reduction in the design's cost or weight. This is in contrast to a decision-making situation in which all performances can be converted into monetary units; in the latter case, two goals can always be traded, or bought, off.

One axiom that is *not* necessary for design-compatible aggregation functions is an axiom of strict monotonicity, and such a requirement would be incompatible with annihilation. The non-compensating function *min* is an example of a function that fulfills all of the axioms of the $\mathrm{M_oI}$ and is not strictly monotonic.

The axioms of idempotency and annihilation set the $\mathrm{M_oI}$ apart from other multi-attribute decision making systems. The reader is referred to [10] for a more detailed discussion of the motivation for this particular set of axioms. This paper posits the axioms of the $\mathrm{M_oI}$ as reasonable for design, and asks what can be said about the functions that satisfy these axioms. Such functions will be known as *design-appropriate*. This paper does not directly address those decision-making problems, outside of the field of engineering design, for which these axioms do not supply a reasonable model, though many of the ideas discussed here will be relevant to other non-design MADM schemes.

3. Fuzzy Multi Attribute Decision Making

In their recent book on the subject, Chen and Hwang [4] identify 18 fuzzy MADM methods, which they systematically classify into eight categories: simple additive weighting methods, the Analytic Hierarchy Process (AHP), the Conjunction/Disjunction method, MAUF, the General MADM method, the outranking method, maximin, and their own proposed MADM method. In their survey, Chen and Hwang do not draw the distinction observed by Zimmermann [24] between continuous Multi Objective Decision Making (MODM) problems and discrete Multi Attribute Decision Making (MADM) problems. This paper shall follow Chen and Hwang and use MADM to refer to the general problem, whether continuous or discrete.

Several of the methods surveyed by Chen and Hwang are similar to the M$_o$I. In addition, the application of utility theory [7] to decision problems bears some similarity to the M$_o$I and to the methods listed above. The possible application of utility theory to engineering design has been considered previously, and shown to be problematic [12]. Matrix methods such as QFD [6] and Pugh charts [13] also support decision making by simple additive aggregation over several requirements.

Aggregation operators are important in all MADM methods, from the most formal to the most casual. The arithmetic mean or weighted sum is popular in matrix methods and elsewhere, as it is simple to calculate. The *min* enjoys considerable popularity as well. Chen and Hwang provide an overview of commonly used aggregation operators; the *min* and the product operators presently in use in the M$_o$I appear in their list, as do weighted sums. However, the general weighted means discussed in this paper do not.

4. Weighted Means

Fuzzy set researchers have productively applied the study of functional equations [1] to explore t-norms and t-conorms. This section applies the same general approach to design-appropriate aggregation functions: an intuitively reasonable set of axioms is translated into a set of functional equations, and these equations are then solved.

A promising class of functions is the class of weighted means. The properties that define weighted means are listed in Table 6.2. While weighted means are defined here as functions of two arguments, they can be extended to several arguments.

The properties of the weighted mean include all of the properties of design-appropriate aggregation functions with the exception of annihilation; a comparison of these properties with the axioms of the M$_o$I shows that **any weighted mean that satisfies annihilation is design-appropriate.** The properties of the weighted mean also include conditions that are not explicitly design axioms; nevertheless, these conditions are consistent with the axioms of the M$_o$I. The bisymmetry condition is a surrogate for commutativity and associativity, and assures that $\mathcal{P}$ can be consistently defined for more than two arguments. Weighted means are strictly monotonic, which is a stronger condition than the monotonicity of the design axioms. **Any strictly monotonic design-appropriate aggregation function must be a weighted mean.** There are design-appropriate functions that are not weighted means, since they are monotonic but fail to satisfy strict monotonicity. Such operators are often

Idempotency: $\mathcal{P}((\mu,\omega_1),(\mu,\omega_2)) = \mu \quad \forall\, \mu,\omega_1,\omega_2$
Internality: $\exists\, \mu_a < \mu_b$ such that $\forall\, \omega_1,\omega_2 > 0$ $\mu_a = \mathcal{P}((\mu_a,1),(\mu_b,0)) < \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2)) < \mathcal{P}((\mu_a,0)(\mu_b,1)) = \mu_b$
Homogeneity of weights: $\mathcal{P}((\mu_a,\omega_1 t),(\mu_b,\omega_2 t)) = \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2))$ $\forall\, \omega_1,\omega_2 \geq 0;\; \omega_1+\omega_2, t > 0$
Bisymmetry: $\mathcal{P}[(\mathcal{P}[(\mu_1,\omega_1),(\mu_2,\omega_2)],\omega_1+\omega_2),(\mathcal{P}[(\mu_3,\omega_3),(\mu_4,\omega_4)],\omega_3+\omega_4)]$ $= \mathcal{P}[(\mathcal{P}[(\mu_1,\omega_1),(\mu_3,\omega_3)],\omega_1+\omega_3),(\mathcal{P}[(\mu_2,\omega_2),(\mu_4,\omega_4)],\omega_2+\omega_4)]$
Increasing in weights: $\mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2)) < \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_3))$ for $\omega_2 < \omega_3 \quad (\mu_a < \mu_b)$
Increasing in variables: $\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2)) < \mathcal{P}((\mu_1,\omega_1),(\mu_3,\omega_2))$ for $\mu_2 < \mu_3, \;\; \omega_2 \neq 0$

Table 6.2 Properties of the Weighted Mean

conditional rather than algebraic; the *min* is but one example. The class of weighted means will not encompass any of these aggregation functions that are only weakly monotonic. However, we shall see that the weak monotonic operators presently used for design can be approximated arbitrarily closely by strictly monotonic operators.

The structure of the class of weighted means is described completely in the following theorem, proven in [1]:

Theorem 1 *The properties of the weighted mean are necessary and sufficient for the function* $\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2))$ *to be of the form*

$$\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2)) = f\left(\frac{\omega_1 f^{-1}(\mu_1)+\omega_2 f^{-1}(\mu_2)}{\omega_1+\omega_2}\right)$$

where $\mu_a \leq \mu_1,\mu_2 \leq \mu_b$; $\omega_1,\omega_2 \geq 0$; $\omega_1+\omega_2 > 0$ *and* f *is a strictly monotonic, continuous function with inverse* f^{-1}.

It follows that any strictly monotonic design-compatible function must have a generating function f. For example, $f(t) = e^t$ (with $\mu_a = 1$ and $\mu_b = e$) generates the familiar weighted product of powers, denoted $\mathcal{P}_\Pi$ and also known as the geometric mean:

$$\mathcal{P}_\Pi((\mu_1,\omega_1),(\mu_2,\omega_2)) = ({\mu_1}^{\omega_1}{\mu_2}^{\omega_2})^{\frac{1}{\omega_1+\omega_2}}$$

This function only satisfies the properties of the weighted mean for $\mu_i > 0$, but it satisfies all of the design axioms, including annihilation, on the closed interval $[0,1]$.

A parameterized family of equations of particular interest for design is generated by the functions $f(t) = t^{\frac{1}{s}}$, where s is a real number. The aggregation function so generated is

$$\mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}}$$

Note that a weighted mean satisfies annihilation if and only if $f^{-1}(0)$ is unbounded for that function. For $s < 0$, $f^{-1}(t) = t^s$ is unbounded at $t = 0$ and $\mathcal{P}_s$ satisfies annihilation. Similarly, $\mathcal{P}_\Pi$ satisfies annihilation, as $f^{-1}(t) = \ln(t)$ is unbounded at $t = 0$. Figure 6.1 shows the behavior of $\mathcal{P}_s$ for several negative values of the parameter s, and for equal weights ($\omega_1 = \omega_2$). In this graph, $\mu_2 = 0.5$ is fixed and μ_1 varies from 0 to 1 along the x-axis. It is graphically evident, and easily shown analytically, that $\mathcal{P}_0$ is identical to the weighted product of powers $\mathcal{P}_\Pi$. Thus the generating function for the geometric mean is not unique. Furthermore, as s tends to $-\infty$, $\mathcal{P}_s$ tends to $\min(\mu_1,\mu_2)$, regardless of the weights.

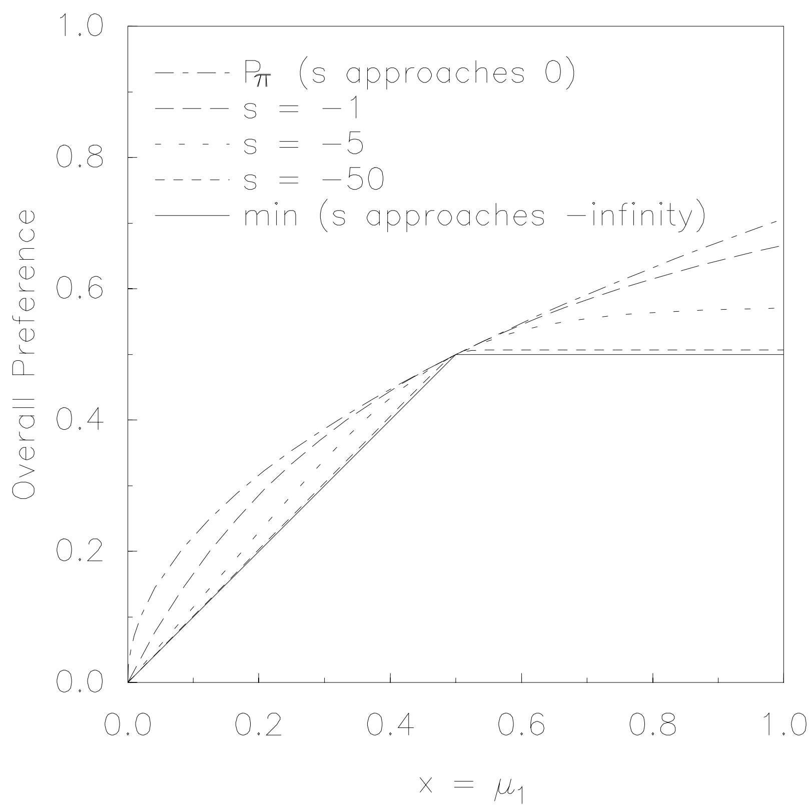

Figure 6.1 Functions between *min* and $\mathcal{P}_{\Pi}$

Claim 1 *$\mathcal{P}_0$ is identical to the weighted product of powers $\mathcal{P}_\Pi$:*

$$\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \mathcal{P}_\Pi((\mu_1,\omega_1),(\mu_2,\omega_2))$$

Proof 1

$$\begin{aligned} &\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) \\ &= \lim_{s\to 0} \left(\frac{\omega_1 {\mu_1}^s + \omega_2 {\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}} \\ &= \exp^{\lim_{s\to 0} \ln\left(\frac{\omega_1 {\mu_1}^s + \omega_2 {\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}}} \\ &= \exp^{\lim_{s\to 0} \frac{1}{s} \ln\left(\frac{\omega_1 {\mu_1}^s + \omega_2 {\mu_2}^s}{\omega_1+\omega_2}\right)} \end{aligned} \tag{6.1}$$

Note that, by the definition of the derivative,

$$\begin{aligned} &\lim_{s\to 0} \frac{1}{s} \ln\left(\frac{\omega_1 {\mu_1}^s + \omega_2 {\mu_2}^s}{\omega_1+\omega_2}\right) \\ &= \frac{d}{ds}\left[\ln\left(\frac{\omega_1 {\mu_1}^s + \omega_2 {\mu_2}^s}{\omega_1+\omega_2}\right)\right]_{s=0} \\ &= \frac{\omega_1 \ln \mu_1 + \omega_2 \ln \mu_2}{\omega_1+\omega_2} \end{aligned}$$

Thus, proceeding from (1), it follows that

$$\begin{aligned} &\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) \\ &= \exp^{\frac{\omega_1 \ln \mu_1 + \omega_2 \ln \mu_2}{\omega_1+\omega_2}} \\ &= ({\mu_1}^{\omega_1} {\mu_2}^{\omega_2})^{\frac{1}{\omega_1+\omega_2}} \\ &= \mathcal{P}_\Pi((\mu_1,\omega_1),(\mu_2,\omega_2)) \end{aligned}$$

which proves the claim.

In light of the preceding claim, the geometric mean $\mathcal{P}_\Pi$ will also be referred to as $\mathcal{P}_0$.

Claim 2 *As $s \to -\infty$, $\mathcal{P}_s$ tends to min:*

$$\lim_{s\to -\infty} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \min(\mu_1,\mu_2)$$

Proof 2

$$
\begin{aligned}
&\lim_{s\to-\infty}\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}} \\
&= \lim_{s\to-\infty}\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{\frac{1}{s}}\left({\mu_1}^s+\frac{\omega_2}{\omega_1}{\mu_2}^s\right)^{\frac{1}{s}} \\
&= \lim_{t\to+\infty}\left(\frac{1}{{\mu_1}^t}+\frac{\omega_2}{\omega_1{\mu_2}^t}\right)^{\frac{1}{-t}} \\
&= \lim_{t\to+\infty}\left(\frac{{\mu_2}^t}{\left(\frac{\mu_2}{\mu_1}\right)^t+\frac{\omega_2}{\omega_1}}\right)^{\frac{1}{t}} \\
&= \lim_{t\to+\infty}\frac{\mu_2}{\left(\left(\frac{\mu_2}{\mu_1}\right)^t+\frac{\omega_2}{\omega_1}\right)^{\frac{1}{t}}}
\end{aligned}
$$

Note that if $\mu_2 \le \mu_1$, then the denominator tends to one and the limit is μ_2; if $\mu_1 < \mu_2$, then the denominator tends to $\frac{\mu_2}{\mu_1}$ and the limit is μ_1. This proves the claim.

Thus the two functions originally proposed as aggregation functions for the M_oI, the *min* and the product of powers, turn out to be two limiting cases of a parameterized family of weighted means. Each $\mathcal{P}_s$, $s \le 0$, models a point on a continuum of trade-off strategies between the original non-compensating and compensating functions.

5. Application to design

In multi-attribute decision making, aggregation functions should provide a useful, justifiable model of the design decision process. The parameterized family $\mathcal{P}_s$ provides a continuum of weighted means between the two existing functions of the M_oI. While this family of functions is useful for design decision-making, it is not exhaustive. Other generating functions give rise to other aggregation functions that satisfy all the axioms of the M_oI but behave differently from any of the $\mathcal{P}_s$.

The *min* is the least compensating possible design-appropriate function. No design aggregation function can take on values less than the *min* at any point in a design space. The *min* function defines a boundary not only of a certain family of weighted means, but also of design-compatible functions in general. The product of powers $\mathcal{P}_\Pi$ is a pivotal example among weighted means, but it is not so clear that it is maximal among design-compatible functions. Indeed, a maximal design-compatible function is problematic: such a function $\mathcal{P}_{\max}$ would satisfy

$$\mathcal{P}_{\max}(0, \mu) = 0 \quad \forall\, \mu$$

but also satisfy

$$\mathcal{P}_{\max}(\epsilon, \mu) = \mu \quad \forall\, \mu > \epsilon > 0$$

for all non-zero weights. $\mathcal{P}_{\max}$ so defined takes on the largest possible value while satisfying both idempotency and annihilation, but it fails another design axiom: it is discontinuous at zero. It is clear that there is no maximal design function corresponding to the *min*.

It was noted above that the arithmetic mean is an aggregation function commonly used in multi-attribute decision making. Yet the arithmetic mean does not satisfy all of the axioms of the MoI, as it fails annihilation. The arithmetic mean is the aggregation function $\mathcal{P}_1$, and allows goals to compensate more strongly than the geometric mean. Indeed, $\mathcal{P}_s$, for $s > 0$, always fails annihilation, and the level of compensation between goals increases with s all the way to $\mathcal{P}_{+\infty} =$ *max*. If the arithmetic mean is only chosen for computational simplicity, then its use must be questioned.

The family of weighted means that formed a continuum between the *min* and $\mathcal{P}_\Pi$ was found by varying the parameter s in $\mathcal{P}_s$ between $-\infty$ and 0. It is interesting to examine what happens if the parameter is varied between 0 and $+\infty$. In this case a family of aggregation functions between $\mathcal{P}_0 = \mathcal{P}_\Pi$ and $\mathcal{P}_\infty =$ *max* is generated. These functions do not satisfy annihilation, so they do not appear to be appropriate for design. As long as no preference approaches zero, however, these functions satisfy all the axioms of the MoI. Figure 6.2 shows $\mathcal{P}_s$ plotted for several positive values of s. As $s \to \infty$, $\mathcal{P}_s \to$ *max*. The "multi-linear" function of utility theory is also shown, though it fails idempotency as well as annihilation.

In practice, the MoI is always implemented using discrete functions. This is partly an artifact of computer implementation, but more fundamentally arises from the fact that some changes in preference are too small to be distinguished. A designer does not actually specify a continuous preference function on the interval $[0, 1]$, but rather gives the values for each variable on several different α-cuts [2]. For example, for a particular design variable, the designer may specify which values correspond to $\mu = 0$, $\mu = 0.25$, $\mu = 0.5$, $\mu = 0.75$, and $\mu = 1$.

The discontinuous manner in which preferences are specified provides some justification for allowing the use of $\mathcal{P}_s$ with $s > 0$, or even the *max* operator, as an aggregation function when all of the preferences achieve some level. This seems to model the design process, as well: when all attributes are performing to some acceptable standard, a designer may choose to allow the highest pref-

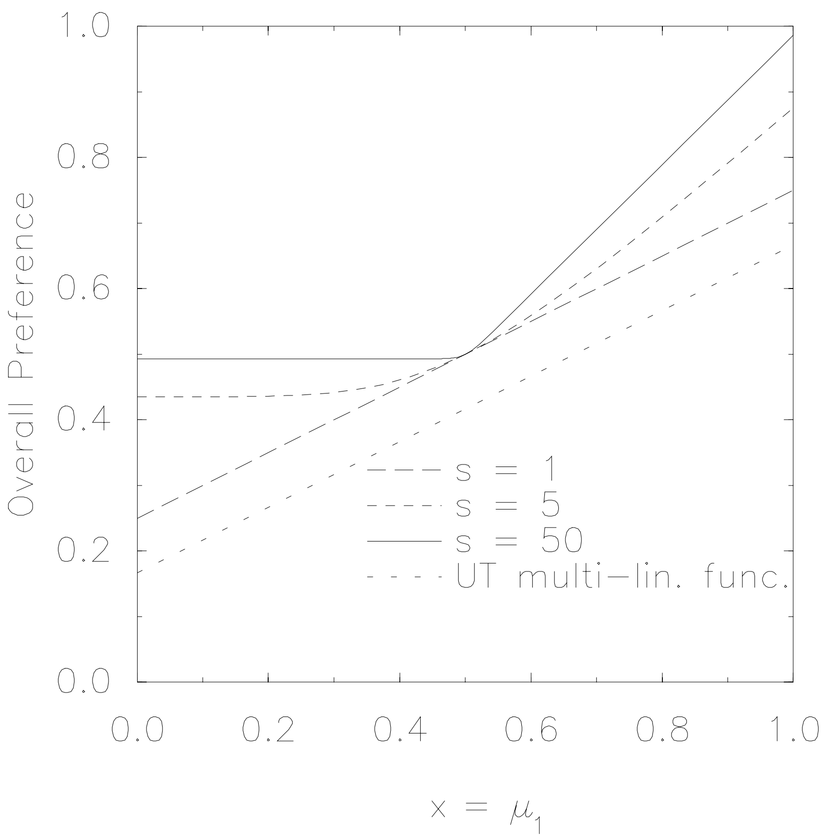

Figure 6.2 Functions that exceed $\mathcal{P}_{\Pi}$

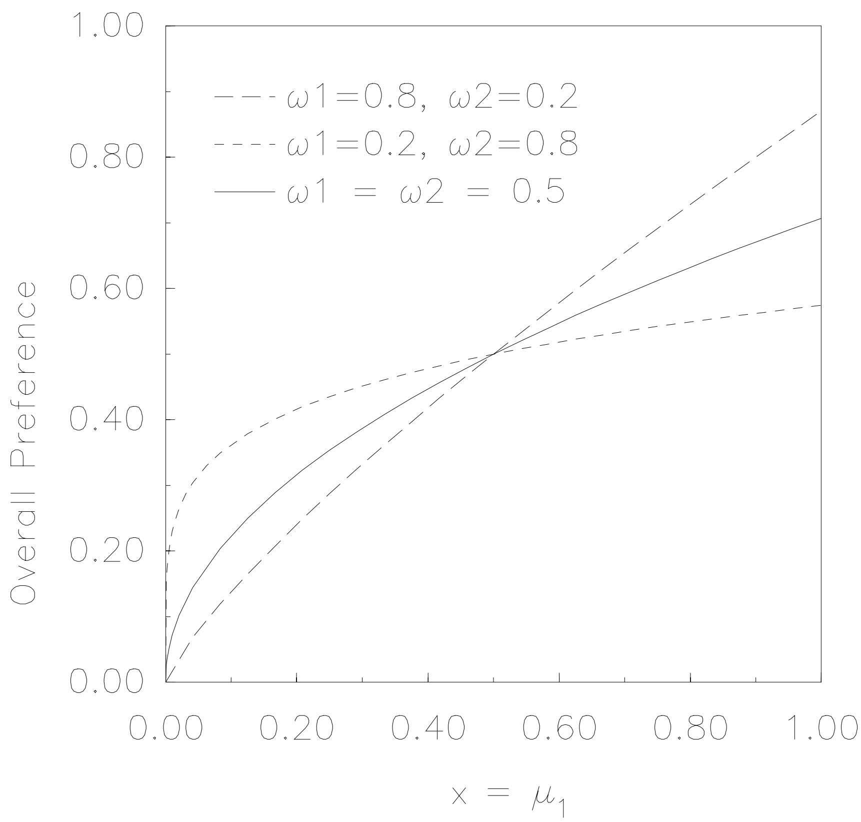

Figure 6.3 The product of powers, with various weights

erences great importance. Unacceptable performance on a single objective, however, can scuttle the design, so annihilation is still satisfied.

Theorem 1 provides a technical justification for the use of the family of operators between $\mathcal{P}_0$ and $\mathcal{P}_{+\infty}$. If it can be assumed that preferences less than some small ϵ, say 0.1, are not relevant to the designer, then the theorem indicates that there is a continuous aggregation function that satisfies $\mathcal{P}(0, y) = 0$ and

$$\mathcal{P}(x, y) = \mathcal{P}_s(x, y) \ \text{ for } \ x, y \geq \epsilon$$

The theorem guarantees that there is a formal operator that models this level of compensation without violating continuity or any other axiom of the MoI.

Thus trade-offs for all cases of compensation, from none ($s = -\infty$) to fully compensating ($s = 0$) to supercompensating ($s > 0$) are accomodated by the weighted means in Theorem 1.

When weights are unequal, even the relatively non-compensating functions between *min* and $\mathcal{P}_\Pi$ can compensate strongly in some regimes. Figure 6.3 shows $\mathcal{P}_\Pi$ with $\mu_2 = 0.5$ fixed and $\mu_1 = x$ as in the other plots, with three different weighting schemes. Notice that when ω_1 and μ_1 are both small, this function offers an annihilating, but still compensatory trade-off. The influence of weights on $\mathcal{P}_\Pi$ could also be exploited to construct a strongly compensating function.

6. Example

To illustrate the family of aggregation operators outlined in this paper, consider example 12-10 from Prof. Zimmermann's textbook [24]. This example is originally presented in the text as a MODM problem, and solved as a fuzzy linear programming problem with continuous variables. In the expression of the problem, however, the variables are quantities of two products to be produced, and it would be natural to assume that they can only take on integer values. Thus the problem can be thought of as a (discrete) MADM problem.

The example involves a company that produces two products, which yield different returns in profit and balance of trade. (Product 1 yields $2 profit but requires $1 in imports; product 2 can be exported for $2 revenue but makes only $1 profit.) The problem is to decide on a "best" production schedule to achieve high profits and a favorable balance of trade. The production schedule is subject to capacity constraints and is modeled by Prof. Zimmermann as follows:

$$\text{“maximize”} \qquad z(x) = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

subject to

$$\begin{aligned} -x_1 + 3x_2 &\leq 21 \\ x_1 + 3x_2 &\leq 27 \end{aligned}$$

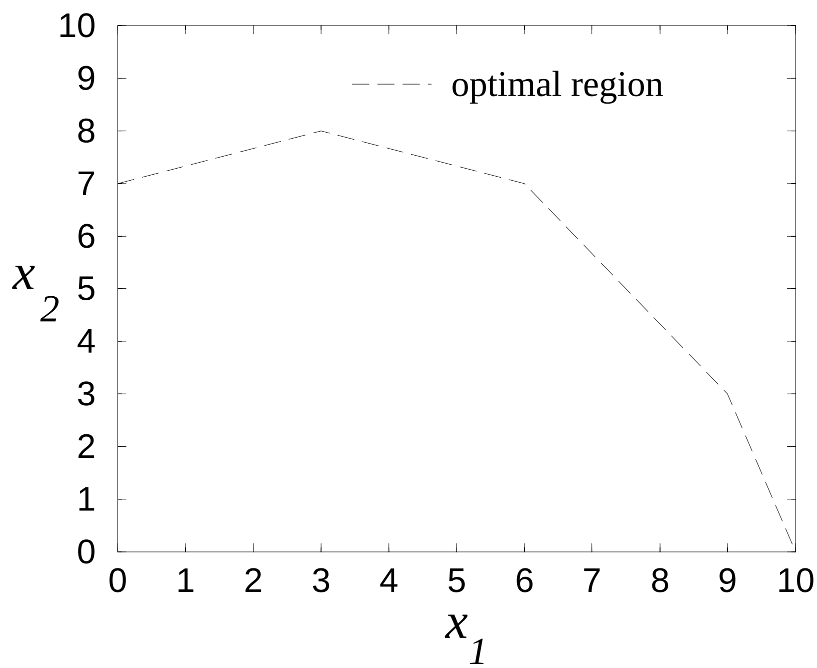

Figure 6.4 Decision Space with Optimal Region

$$
\begin{aligned}
4x_1 + 3x_2 &\leq 45 \\
3x_1 + x_2 &\leq 30 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Prof. Zimmermann shows a plot of the decision space with a region of optimal values, similar to the one shown in Figure 6.4. There are seven undominated points in the decision space, which are shown in Table 6.3.

It is clear that this problem is not simply an exercise in mathematical programming. Three important questions remain. First, the decision maker must specify what it means to achieve the two individual goals (high profit, favorable balance of trade). Second, the relative importance of the goals must be addressed. Third, to what extent should high performance with respect to one goal be allowed to compensate for low performance elsewhere?

The example is solved in the textbook by an application of fuzzy sets that is substantively similar to that used by the M_oI. The first step is to determine a level of satisfaction for each of the two goals, in essence to create the two fuzzy sets "Decisions that satisfy the profit goal" and "Decisions that satisfy the balance of trade goal." What the M_oI would call *preference* the textbook refers to as *level of satisfaction*. The preference or satisfaction for the performance

(x_1, x_2)	z_1	z_2
(0,7)	14	7
(3,8)	13	14
(4,7)	10	15
(5,7)	9	17
(6,7)	8	19
(8,4)	0	20
(9,3)	-3	21

Table 6.3 Undominated points in the decision space.

on balance of trade increases linearly from $\mu_1(x) = 0$ at $z_1(x) = -3$ to $\mu_1(x) = 1$ at $z_1(x) = 14$. The preference for profit increases linearly from $\mu_2(x) = 0$ at $z_2(x) = 7$ to $\mu_2(x) = 1$ at $z_2(x) = 21$. These preferences are generated in the textbook with reference to the values listed in Table 6.3; in the general application of the M_oI, the memberships μ_i may be specified using a different process. In any event, the given fuzzy sets "Decisions that satisfy the objectives", with memberships ranging from 0 to 1 throughout the decision space, are valid preference functions for the M_oI.

The problem of relative importance of the two goals does not come into play in this problem, and it can be assumed that the two goals are equally weighted. The aggregation functions presented in this paper can model preferential weighting of an arbitrary number of goals. However, the choice of an aggregation function, which encapsulates the decision of how much high performance on on attribute is to compensate for low performance on the other, remains.

If the decision problem is treated as a fuzzy linear programming problem, as in the text, the aggregation function used is the *min*. The *min* is "natural" for ease of computation, but not necessarily natural for the decision. When the problem is solved using the *min* operator, as shown in Figure 6.5, the maximum degree of "overall satisfaction" is given by the point $x_0 = (5.03, 7.32)$, with $\mu_o = 0.74$. Among the integer choices available, $x_0 = (5, 7)$ is the best, with $\mu_o = 0.71$. This corresponds to the solution given by the M_oI when it is determined that the problem is non-compensating, *i.e.*, the overall performance is limited by the lowest performance of all attributes.

In many situations, the overall performance of a design, or the general attractiveness of a decision, is not limited by the lowest performance among the attributes. For each problem, there is a level of compensation that is appropriate. The selection of the appropriate level of aggregation is one question; how to model it is another. This paper focuses on the second question and defers

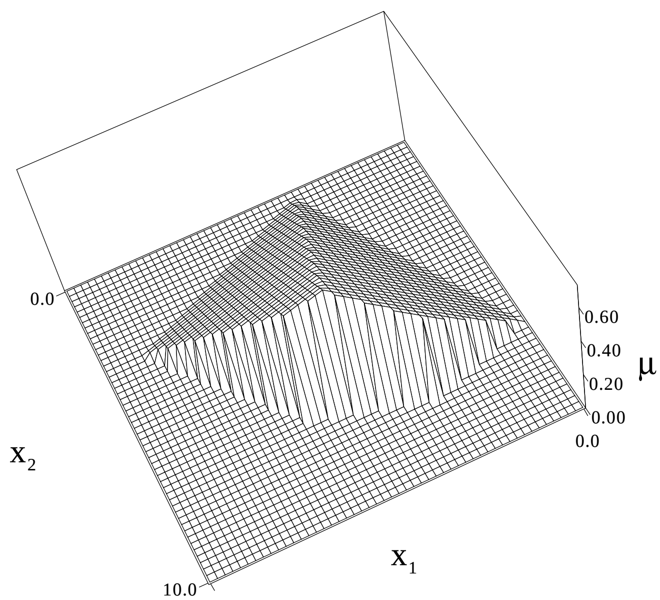

Figure 6.5 Decision surface with minimum operator

the discussion of the first. However, it shall be seen that different levels of compensation lead to different decisions.

The MoI has used a weighted product of powers for problems that demand a higher level of compensation than is afforded by the minimum function. The point with highest combined preference is $x_1 = (5.70, 7.10)$ with $\mu_o = 0.75$, as shown in Figure 6.6. The point of highest preference is not far from the point of highest preference achieved with the *min* operator. However, if one considers only the discrete points with integer values (if the company cannot manufacture 5.70 of a product), the highest performing point is $x_1 = (6, 7)$ (with $\mu_o = 0.74$). A different level of compensation among goals leads to a different decision.

The application of the entire family of of aggregation functions $\mathcal{P}_s$ to this example problem shows that there are at least three optimal points, each suitable over a range of values of the parameter s, and thus over a range of levels of compensation of goals. In addition, the point (1,7), though dominated by the point (3,8), approaches it asymptotically in preference as the level of compensation approaches the maximum. Figure 6.7 shows the overall preferences for all of these "optimal points" calculated using $\mathcal{P}_s$, with the parameter s ranging from -10 to 10. When $\mathcal{P}_s = \min$ ($s \to -\infty$), the point with the highest pref-

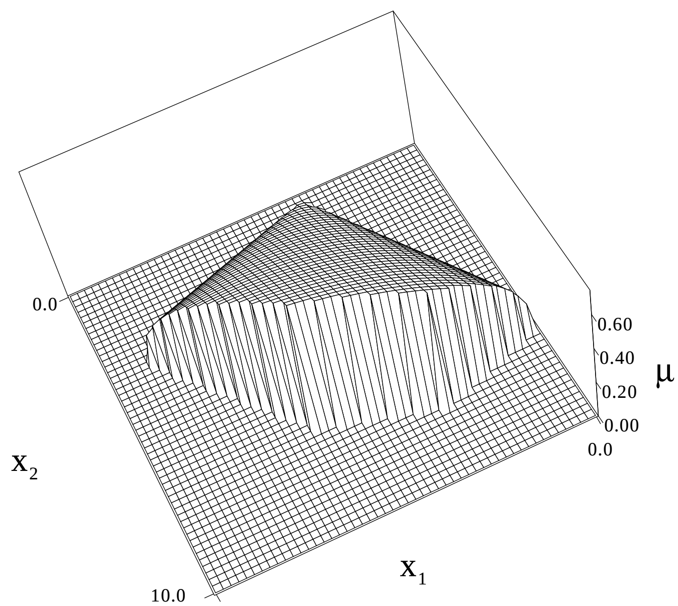

Figure 6.6 Decision surface with product of powers operator

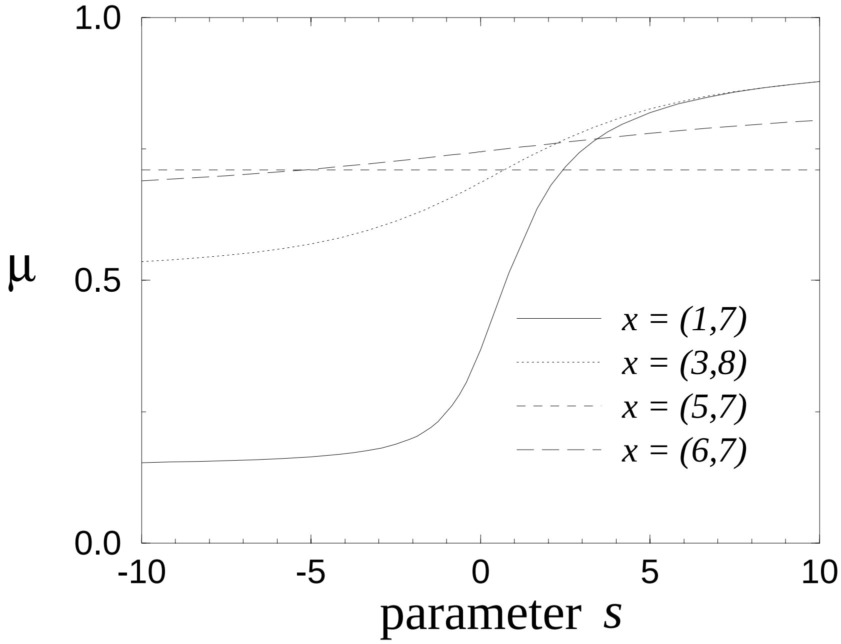

Figure 6.7 Optimal Points Varying with Parameter s

erence is $x = (5,7)$, the textbook answer. As s grows and $\mathcal{P}_s$ becomes more compensating, the preference for other points grows stronger. As s crosses -5.25, $x = (6,7)$ overtakes $x = (5,7)$ and remains the most preferred point through approximately $s = 2.25$. This region includes two common aggregation functions: the weighted product of powers or geometric mean ($s = 0$), and the classical weighted sum or arithmetic mean ($s = 1$). Values of s greater than 2.25, corresponding to even stronger levels of compensation, lead to an overall preference for the point (3,8), and as $s \to +\infty$, the preference for the dominated point $x = (1,7)$ approaches the preference for $x = (3,8)$ asymptotically. The limits $s = 10$ and $s = -10$ were chosen to show the full range of behavior of this problem; values of s outside this range are certainly permissible.

The functions discussed in this paper provide models for a continuum of trade-offs ranging from the non-compensating *min* to the compensating $\mathcal{P}_\Pi$ all the way to the *max* operator. Instead of two aggregation functions, there is a parameterized family of functions ranging from the *min* to $\mathcal{P}_\Pi$, and another from $\mathcal{P}_\Pi$ to the *max*.

Conclusion

This paper has discussed the selection of an aggregation function for Multi Attribute Decision Making. The problem was investigated within the context of the Method of Imprecision, a formal system, based on the mathematics of fuzzy sets, for the representation and manipulation of imprecise design information through the specification of preferences on design and performance variables. The MoI casts the preliminary design decision problem as a MADM problem, and uses different aggregation functions to formally model different trade-off strategies. The class of functions appropriate for the aggregation of these preferences has been explored in this paper. While they are directly applicable to decision making in engineering design, the results of this paper are also relevant to other MADM schemes.

This paper has presented a complete characterization of aggregation functions that satisfy the axioms of the MoI. The class of functional equations known as quasi-linear weighted means was shown to be crucial. It was demonstrated that any strictly monotonic design-compatible aggregation function is generated by a generating function as detailed in Theorem 1. The conditional operators in use, while not weighted means, were shown to be limits of sequences of such functions. A parameterized family of functions was detailed, spanning two continua of possible design strategies, one between the non-compensating *min* and the compensating $\mathcal{P}_\Pi$, the two original aggregation functions of the MoI, and one between $\mathcal{P}_\Pi$ and the *max*.

Many MADM systems use aggregation functions, such as the arithmetic mean, that compensate between goals more aggressively than the existing functions of the MoI. These highly compensating functions may seem to be in conflict with the axioms of the MoI. This paper has assessed the possibility of using these common aggregation functions in design decision-making problems.

There are an infinite number of aggregation operators that are suitable for engineering design. In particular, the parameterized family $\mathcal{P}_s$ is a range of functions that models a broad spectrum of design decision-making situations, with the parameter s indicating the degree of compensation permitted among performance criteria. The appropriate choice of the parameter s is problem-specific and is a matter for further investigation.

The use of an aggregation function in any MADM system may be justified on empirical as well as rational grounds. It is not within the scope of this paper to provide empirical studies of designers' decision making behavior. The development here has focused on a rational basis for the choice of an aggregation function. More empirical studies are needed to confirm in practice the results in this paper.

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