AGGREGATION FUNCTIONS FOR ENGINEERING DESIGN TRADE-OFFS

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ABSTRACT

The *Method of Imprecision* (MoI) is a formal theory for the manipulation of preliminary design information that represents preferences among design alternatives with the mathematics of fuzzy sets. Using the MoI, different design trade-off strategies can be applied. To date, two aggregation functions have been developed for the MoI, one representing a compensating strategy and one a non-compensating strategy. Other research on aggregation functions on fuzzy sets has focused on two classes of functions that are not suitable for engineering design. The general restrictions on design-appropriate aggregation functions are discussed, and a family of functions ranging from the non-compensating *min* to the compensating *product of powers* is presented. An application to preliminary engineering design is given.

Introduction

Preliminary design decisions are among the most important in engineering. It is when the details of a design are still unknown, when the design is *imprecise*, that the most costly and important decisions are made (Whitney, 1988). Despite the importance of preliminary design, there are a limited number of formal tools available to the engineer for representing and manipulating imprecise design information. One such tool is the *Method of Imprecision* (MoI) (Wood and Antonsson, 1989), a formal method for representing and manipulating uncertainty in engineering design employing the mathematics of fuzzy sets. It has been shown previously that the MoI can be used to combine design information using different trade-off strategies (Otto and Antonsson, 1991). Specifically, the designer can combine preferences in a compensating or non-compensating manner, using one of two aggregation functions. The MoI can handle importance weightings using these two trade-off strategies (Law and Antonsson, 1995a; Otto and Antonsson, 1991), and different attributes can be combined hierarchically in hybrid strategies. Research applying both trade-off strategies to industrial applications is ongoing (Law and Antonsson, 1994).

A class of aggregation functions on fuzzy sets called t-norms and t-conorms has been studied in some detail (Dubois and Prade, 1985; Roychowdhury and Wang, 1994) by researchers of fuzzy sets. Unfortunately, these functions do not satisfy the axioms of the MoI, as will be explained below. Much less research has been devoted to mixed connectives, those functions that are suitable for engineering design. This paper takes the general approach that fuzzy set researchers have applied in exploring t-norms and t-conorms and applies it to the problem of design-appropriate aggregation functions.

After a brief review of the MoI, aggregation functions will be discussed in more detail. A continuum of functions between the compensating and the non-compensating will be presented. An example of an application to preliminary engineering from an earlier paper will be revisited. Where two aggregation functions were used before, the full range of aggregation functions will be applied.

Preliminaries: The Method of Imprecision

Detailed definitions for the Method of Imprecision are presented elsewhere (Law and Antonsson, 1995b). An example of an application to preliminary design from the original paper on trade-off strategies (Otto and Antonsson, 1991) will serve to review the important definitions. This example is a computationally tractable analog of some industrial applications presently being investigated.

The example is of a preliminary design task, a selection between a special-purpose mechanism and a robotic arm for

Criterion: Ease to Satisfy	Mechanism (μ_i, ω_i)	Robot (μ_i, ω_i)
Quantity rate	$(1.0, \frac{4}{27})$	$(0.5, \frac{4}{27})$
Operator safety	$(0.4, \frac{4}{27})$	$(0.5, \frac{4}{27})$
Development cost	$(0.4, \frac{5}{27})$	$(0.7, \frac{5}{27})$
Production reliability	$(0.8, \frac{5}{27})$	$(0.5, \frac{5}{27})$
Size constraints	$(0.9, \frac{2}{27})$	$(0.7, \frac{2}{27})$
Design by production time	$(0.5, \frac{3}{27})$	$(0.8, \frac{3}{27})$
Production quality	$(1.0, \frac{4}{27})$	$(0.9, \frac{4}{27})$

TABLE 1. Imprecise designer rankings.

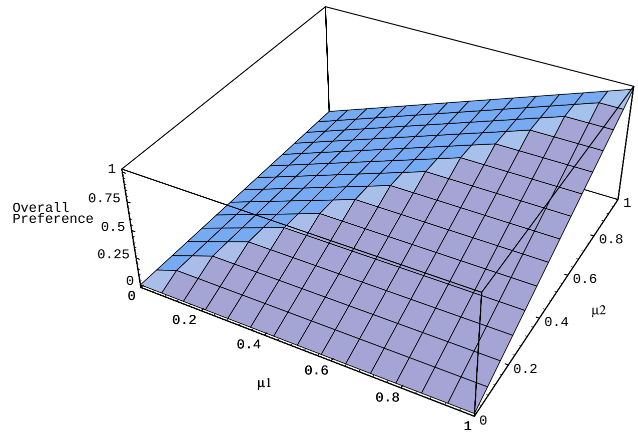

FIGURE 1. 3D plot of $\min(\mu_1, \mu_2)$

a manufacturing production line. Designers assign preferences $\mu_i \in [0,1]$ and importance weightings ω_i to seven criteria as shown in Table 1. Since this example is of a choice between discrete alternatives, the preferences are crisp numbers; in general, each preference will be expressed as a fuzzy set over a range of values for each design variable (in this case, each criterion). A preference of $\mu_i = 0$ indicates that the value is completely unacceptable and $\mu_i = 1$ indicates an ideal value. Designers also assign importance weightings ω_i to the attributes. In this example the weights are normalized to sum to one; later we shall see that this is not necessary.

It is the designer's task to choose the best overall design among the possible alternatives. When different candidate designs have different strengths, the designer must make trade-offs among the design attributes. These trade-offs are commonly made informally. The nature of the trade-off depends on the particular design problem: in some cases the designer may need to guarantee that all attributes satisfy a minimal performance criterion, while in other cases strong performance in one aspect may make up for poorer performance elsewhere. In the MoI, the preferences (with their respective weights) for the variables are combined into a single overall preference μ_o using an *aggregation function* $\mathcal{P}$:

$$\mu_o = \mathcal{P}((\mu_1, \omega_1), \cdots, (\mu_n, \omega_n))$$

If it is convenient to ignore the weights, the function can be written simply

$$\mu_o = \mathcal{P}(\mu_1, \cdots, \mu_n)$$

Two aggregation functions have been used to represent two trade-off strategies: when the design variables are to be combined in a non-compensating manner, in which the overall performance of the design is limited by the worst-performing attribute, the simple minimum is used:

$$\mathcal{P}((\mu_1, \omega_1), \cdots, (\mu_n, \omega_n)) = \min(\mu_1, \cdots, \mu_n)$$

For designs that require a compensating trade-off strategy, in which a high preference for one attribute may make up for a lower preference for another, the *weighted product of powers*, or $\mathcal{P}_\Pi$, is used:

$$\mathcal{P}_\Pi((\mu_1, \omega_1), \cdots, (\mu_n, \omega_n)) = ({\mu_1}^{\omega_1} \cdots {\mu_n}^{\omega_n})^{(\frac{1}{\omega_1 + \cdots + \omega_n})}$$

The behavior of these aggregation functions can be fully represented by a three-dimensional plot of the combined overall preference μ_o rising above the $\mu_1 - \mu_2$ plane, as shown for $\mathcal{P} = min$ in Figure 1. For comparing different functions, a cross-section of the 3D plot is instructive. Figure 2 shows a cross-section of a three-dimensional plot of both $\min(\mu_1, \mu_2)$ and $\mathcal{P}_\Pi((\mu_1, \omega), (\mu_2, \omega))$ taken at $\mu_2 = 0.5$. (The equal weights ω are assumed to be non-zero.) Because of restrictions on aggregation functions that will be discussed later, both functions deliver the same overall preference at $x = 0$ and $x = 0.5$: $\min(0, 0.5) = \mathcal{P}_\Pi((0, \omega), (0.5, \omega)) = 0$ and $\min(0.5, 0.5) = \mathcal{P}_\Pi((0.5, \omega), (0.5, \omega)) = 0.5$. Elsewhere, combining $\mu_1 = x$ and $\mu_2 = 0.5$ with the compensating function $\mathcal{P}_\Pi$ always gives a higher overall preference than combining them with *min*. Compensating aggregation functions are appropriate for design trade-offs where high preference for one attribute can partially compensate for low preference in another. A similar cross-section could be taken for each fixed value of μ_2; the crossover point would change, but the shape would remain the same. Since the functions are symmetric, the same plots could be generated by fixing μ_1 and allowing μ_2 to vary.

In the example shown earlier, three trade-off strategies were considered: a traditional weighted sum analysis, the non-compensating function *min*, and the compensating function $\mathcal{P}_\Pi$. The weighted sum favored the mechanism, with overall preferences of 0.70 and 0.64. The compensating function $\mathcal{P}_\Pi$ also rated the mechanism best, 0.65 to 0.63. while the *min* preferred the robotic arm, 0.5 to 0.4. Thus a non-compensating trade-off strategy was shown to have a different outcome from a compensating one.

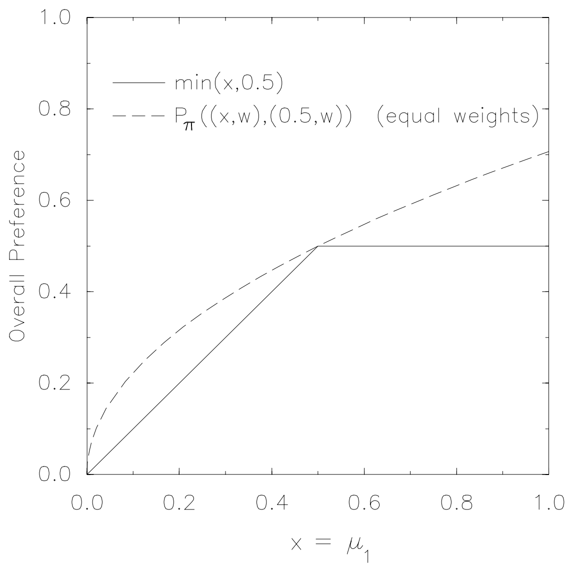

FIGURE 2. Two aggregation functions

The two aggregation functions (*min* and weighted product) model two different trade-off strategies. There may be other possible strategies. The traditional weighted sum seems to be an example of a function that, in this case at least, models a more compensatory trade-off strategy than $\mathcal{P}_\Pi$, and it might be imagined that there are functions that give results between $\mathcal{P}_\Pi$ and *min*. This paper will revisit the conditions that an aggregation function must satisfy to be appropriate for design. It shall be seen that in addition to the two original functions, there is a continuum of functions modeling a continuum of strategies between them, as well as functions that satisfy the conditions for design and exceed $\mathcal{P}_\Pi$ in the amount that they allow a high preference to compensate for a lower one.

Axioms for Design-Appropriate Aggregation Functions

Engineering design as it is commonly practiced is an informal undertaking. In order to place a structure on design, a system must conform to axioms that formally represent an intuitively reasonable model of the design process. The original axioms for aggregation functions for design are discussed in detail by Otto (1992). Their extension to include weights is given in detail in the appendix. The class of possible functions that satisfy these axioms will be investigated below. Three of the axioms are particularly important to the discussion here.

The annihilation axiom is crucial for design, as others have argued (Biegel and Pecht, 1991; Vincent, 1983). It states that if the preference for any one attribute of the design sinks to zero (unaceptable) then the overall preference for the design is zero. For example, given a fixed material, the tensile strength limit cannot be exceeded no matter the reduction in the design's cost or weight.

The idempotency axiom states that if several variables with equal preferences are combined, the overall preference must be the same. Idempotency reflects the intuitive constraint that the overall preference for a design should never exceed the preference of the highest-ranked attribute, nor should it fall below the preference of the lowest-ranked attribute. Idempotency and monotonicity together are equivalent to the condition that $min \le \mathcal{P} \le max$.

Monotonicity states that the overall preference cannot decrease as a result of an increase in the preference for one attribute, and is required for design. Strict monotonicity is not required and is incompatible with annihilation. The non-compensating function *min* is an example of a design-appropriate function that is monotonic but not strictly monotonic.

Connectives on Fuzzy Sets

The MoI is just one formalization that uses fuzzy sets to represent preferences or satisfaction of goals, and not all formalizations have used the same axioms. Research on aggregation functions or connectives for decision-making using fuzzy sets, for example, has focused on extensions of classical intersection and union (Dubois and Prade, 1985). The operators of binary logic have been applied to fuzzy truth values to develop a theory for the aggregation of fuzzy goals (Bellman and Zadeh, 1970). In this system, a decision maker may have one of three attitudes towards goals: a "conjunctive attitude", in which goals must be simultaneously satisfied, modeled by the logical *and* and the intersection of fuzzy sets; the "disjunctive attitude", in which goals are redundant, modeled by the logical *or* and the union of fuzzy sets; and a "compromise attitude", where one goal can be traded off against another. A class of functions called triangular norms or t-norms have been used to represent intersection, and the complementary t-conorms or s-norms to represent union. These functions, and some others, are discussed in detail by Dubois and Prade (1985). T-norms are bounded above by *min*, and it is easy to show that *min* is the only idempotent t-norm. Likewise, t-conorms are bounded below by *max*, which is the only idempotent t-conorm.

Thus the functions most investigated as aggregation functions on fuzzy sets model the logical *and* and the logical *or* and are inappropriate for design, since they fail idempotency or annihilation or both. To obey the axioms of the MoI, the only possible function for modelling a conjunctive attitude is the *min*. No function that can model a disjunctive attitude; the *max* is the most likely, but it fails the axiom of annihilation.

Design-Appropriate Aggregation Functions

This section takes the general approach that other fuzzy set researchers applied in exploring t-norms and conorms and applies it to design-appropriate aggregation functions. The problem is approached from the point of view of functional equations (Aczél, 1966): the set of intuitively reasonable axioms are translated into a set of functional equations, which are then solved.

A promising class of functions is the class of weighted means. The properties that define weighted means are listed in the appendix for functions of two arguments (where each argument is a preference-weight pair). These functions can be extended to several arguments; the details will be discussed below.

The properties of the weighted mean include all of the properties of design-appropriate aggregation functions with the exception of annihilation; any weighted mean that satisfies annihilation can serve as a design aggregation function. The properties of the weighted mean also include several conditions that are not in the design axioms. The bisymmetry condition is a surrogate for commutativity and associativity, and is equivalent to a requirement that $\mathcal{P}$ be consistently defined for more than two arguments. Weighted means are strictly monotonic, which is a stronger condition than the monotonicity of the design axioms. Any strictly monotonic design-appropriate aggregation function must be a weighted mean. There are design-appropriate functions that are not weighted means; the *min* is an example. However, these functions can be approximated arbitrarily closely by (strictly monotonic) weighted means.

The structure of the class of weighted means is described completely in the following theorem, proven in Aczél (1966):

Theorem 1 *The properties of the weighted mean (see Appendix) are necessary and sufficient for the function* $\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2))$ *to be of the form*

$$\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2)) = f\left(\frac{\omega_1 f^{-1}(\mu_1)+\omega_2 f^{-1}(\mu_2)}{\omega_1+\omega_2}\right)$$

where $\mu_a \le \mu_1, \mu_2 \le \mu_b$; $\omega_1, \omega_2 \ge 0$; $\omega_1+\omega_2 > 0$ *and* f *is a strictly monotonic, continuous function with inverse* f^{-1}.

It follows that all strictly monotonic design-compatible functions must have a generating function f. For example, $f(t) = e^t$ (with $\mu_a = 1$ and $\mu_b = e$) generates the familiar weighted product of powers

$$\mathcal{P}_{\prod}((\mu_1,\omega_1),(\mu_2,\omega_2)) = ({\mu_1}^{\omega_1}{\mu_2}^{\omega_2})^{\frac{1}{\omega_1+\omega_2}}$$

also known as the *geometric mean*. This function only satisfies the properties of the weighted mean for $\mu_i > 0$, but it satisfies all of the design axioms, including annihilation, on the closed interval $[0,1]$.

A parameterized family of equations of particular interest for design is generated by the functions $f(t) = t^s$, where s is a real parameter. The aggregation function so generated is

$$\mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}}$$

Note that a weighted mean satisfies annihilation if and only if $f^{-1}(0)$ is unbounded for that function. For $s < 0$,

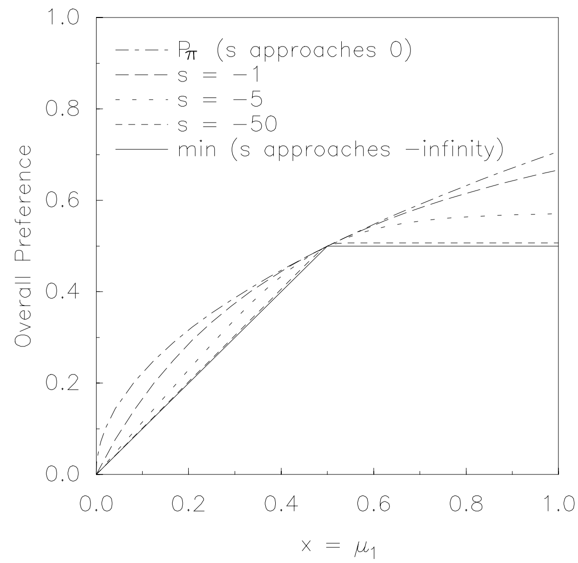

FIGURE 3. Functions between *min* and $\mathcal{P}_{\prod}$

$f^{-1}(t) = t^{\frac{1}{s}}$ is unbounded at $t = 0$ and $\mathcal{P}_s$ satisfies annihilation. Similarly, $\mathcal{P}_{\prod}$ satisfies annihilation, as $f^{-1}(t) = \ln(t)$ is unbounded at $t = 0$. Figure 3 shows the behavior of $\mathcal{P}_s$ for several negative values of the parameter s, and for equal weights ($\omega_1 = \omega_2$). As before, $\mu_2 = 0.5$ is fixed and μ_1 varies from 0 to 1 along the x-axis. It is graphically evident, and easily shown analytically (see Appendix), that $\mathcal{P}_0$ is identical to the weighted product of powers $\mathcal{P}_{\prod}$. As s tends to $-\infty$, $\mathcal{P}_s$ tends to $\min(\mu_1,\mu_2)$ regardless of the weights.

Thus the two functions originally proposed as aggregation functions for the MoI, the *min* and the product of powers, turn out to be two limiting cases of a parameterized family of weighted means. Each $\mathcal{P}_s$, $s \le 0$, models a point on a continuum of trade-off strategies between the original non-compensating and compensating functions.

The *min* defines the axiomatic expression of non-compensating: the overall preference for any design is limited by its least-performing attribute, no matter how well other attributes perform. No design aggregation function can take on values less than the *min* at any point in a design space. Thus the *min* function defines a boundary not only of a certain family of weighted means, but also of design-compatible functions in general. The product of powers $\mathcal{P}_{\prod}$ is a pivotal example among weighted means, but it is not so clear that it is maximal among design-compatible functions. Indeed, a maximal design-compatible function is problematic: such a function $\mathcal{P}_{\max}$ would satisfy

$$\mathcal{P}_{\max}(0,\mu) = 0 \quad \forall\ \mu$$

but also satisfy

$$\mathcal{P}_{\max}(\epsilon,\mu) = \mu \quad \forall\ \mu > \epsilon > 0$$

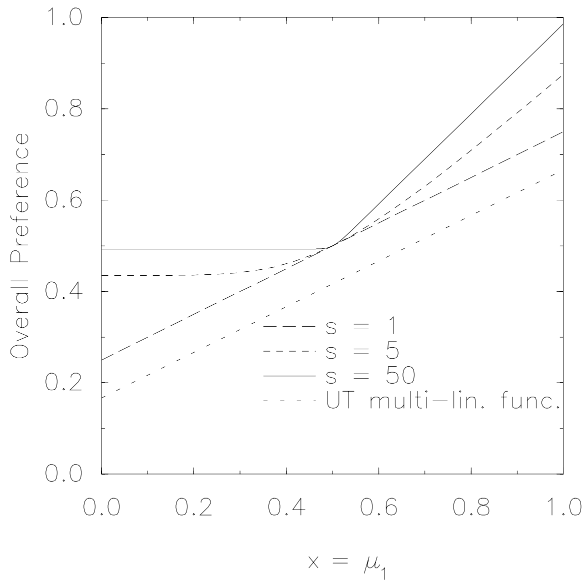

FIGURE 4. Functions that exceed $\mathcal{P}_{\prod}$

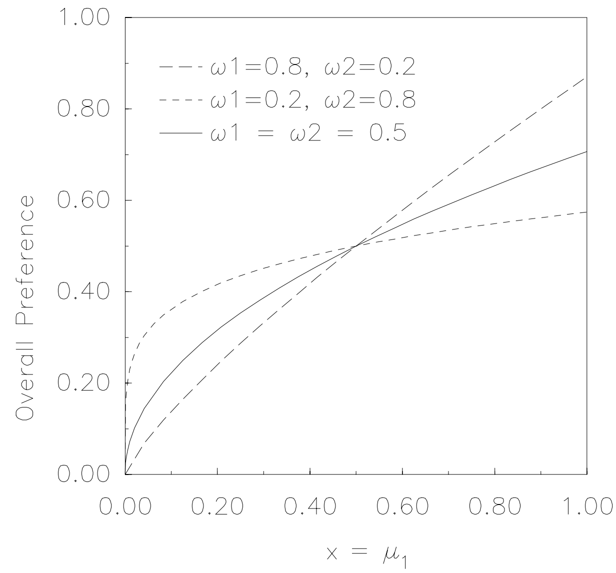

FIGURE 5. The product of powers, with various weights

for all non-zero weights. $\mathcal{P}_{\max}$ so defined takes on the largest possible value while satisfying both idempotency and annihilation, but it fails another design axiom: it is discontinuous at zero. It is clear that there is no maximal design function in the way that there is a minimal function.

The family of weighted means that formed a continuum between the *min* and $\mathcal{P}_{\prod}$ was found by varying the parameter s in $\mathcal{P}_s$ between $-\infty$ and 0. It is interesting to examine what happens if the parameter is varied between 0 and $+\infty$. In this case a family of aggregation functions between $\mathcal{P}_0 = \mathcal{P}_{\prod}$ and $\mathcal{P}_\infty = max$ is generated. These functions do not satisfy annihilation, so they do not appear to be appropriate for design. As long as no preference is zero, however, these functions satisfy all the axioms of the MoI. Figure 4 shows $\mathcal{P}_s$ plotted for several positive values of s. As $s \to \infty$, $\mathcal{P}_s \to max$.

If the designer preferences are taken to represent utilities, then the aggregation functions used by utility theory (Keeney and Raiffa, 1993) can be similarly plotted. The additive utility function ($\mathcal{P}((\mu_1, \omega_1), (\mu_2, \omega_2)) = \omega_1\mu_1 + \omega_2\mu_2$) corresponds to $\mathcal{P}_1$ in Figure 4. The “multi-linear” function, also shown in Figure 4 ($\mathcal{P}((\mu_1, \omega_1), (\mu_2, \omega_2)) = \frac{\mu_1}{3} + \frac{\mu_2}{3} + \frac{\mu_1\mu_2}{3}$, in this case), fails both idempotency and annihilation and is not a member of this parameterized family of weighted means.

In practice, the MoI is always implemented using discrete functions. This is partly an artifact of computer implementation, but more fundamentally arises from the fact that some changes in preference are too small to be distinguished. A designer does not actually specify a continous preference function on the interval $[0, 1]$; she gives the values for each variable on several different α-cuts (Antonsson and Otto, 1995). For example, for a particular design variable, the designer may specify which values correspond to $\mu = 0$, $\mu = 0.25$, $\mu = 0.5$, $\mu = 0.75$, and $\mu = 1$. Preference between the α-cuts is not important, as the differences are too small to matter.

The discontinuous manner in which preferences are specified provides some justification for allowing the use of the *max* operator, or of $\mathcal{P}_s$ with $s > 0$, as an aggregation function when all of the preferences are above some level. This seems to model the design process, as well: when all attributes are performing to some acceptable standard, a designer may choose to allow the highest preferences great importance. The choice of trade-off strategy may depend on the preferences. If it can be assumed that preferences less than some small ϵ, say 0.1, are meaningless to the designer, then Theorem 1 indicates that there is an aggregation function that varies between $\mathcal{P}(0, x) = 0$ and $\mathcal{P}(\epsilon, x) = x$ in a continuous way.

When weights are unequal, even the relatively conservative functions between *min* and $\mathcal{P}_{\prod}$ can compensate strongly in some regimes. Figure 5 shows $\mathcal{P}_{\prod}$ with $\mu_2 = 0.5$ fixed and $\mu_1 = x$ as in the other plots, with three different weighting schemes. Notice that when ω_1 and μ_1 are both small, this function offers an annihilating, but still compensatory tradeoff. The influence of weights on $\mathcal{P}_{\prod}$ could also be exploited to construct a strongly compensating function.

Hierarchical trade-offs

An important requirement of any design-compatible aggregation function is the ability to combine preferences hierarchically. This is equivalent to requiring that an aggregation function be extendible to an arbitrary number of arguments, which reduces in turn to commutativity, associa-

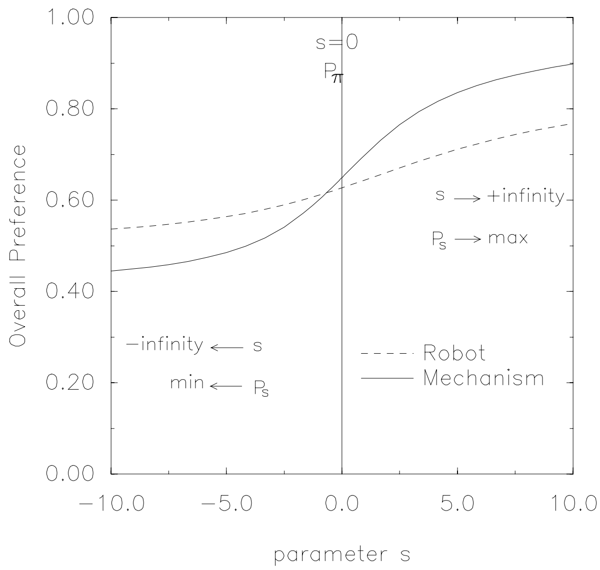

FIGURE 6. Robot Arm vs. Mechanism, various $\mathcal{P}_s$

tivity, and an iterative definition of $\mathcal{P}$ of many arguments. Since weighted means are all associative and commutative, the results for aggregation functions of two arguments extend obviously to aggregation functions of more than two arguments.

Given a set of weighted preferences $(\mu_1, \omega_1), \ldots, (\mu_n, \omega_n)$, define

$$\mathcal{P}((\mu_1, \omega_1),..., (\mu_n, \omega_n)) = f\left(\frac{\omega_1 f^{-1}(\mu_1) + ... + \omega_n f^{-1}(\mu_n)}{\omega_1 + ... + \omega_n}\right)$$

where f is the generator for $\mathcal{P}$. $\mathcal{P}$ is clearly well-defined, and when applied to any n arguments it can be found by successive application to pairs of the arguments. It is important to note that the weights for all arguments should be assigned in advance, and that the weight for $\mathcal{P}$ applied to several arguments must be the sum of the weights of the individual arguments. It is possible, though unnecessary, to normalize at each step.

Example

The functions discussed in this paper provide models for a continuum of trade-offs ranging from the non-compensating *min* to the partially compensating $\mathcal{P}_\Pi$ all the way to the completely compensating *max*.

The example introduced earlier, of the choice between a robotic arm and a mechanism, can now be revisited. Instead of two aggregation functions, there is a parameterized family of functions ranging from the *min* to $\mathcal{P}_\Pi$, and another from $\mathcal{P}_\Pi$ to the *max*. (Since the lowest preference considered is 0.4, annihilation is not an issue.)

Figure 6 shows the overall preferences for the robotic arm and the mechanism calculated using $\mathcal{P}_s$, with the parameter s ranging from -10 to 10. The previous results are included here. At $s = 0$, $\mathcal{P}_s = \mathcal{P}_\Pi$, and the mechanism is favored slightly over the robot. As s grows and $\mathcal{P}_s$ becomes more compensating, the preference for the mechanism grows stronger. At sufficiently small s, the robot is preferred; there is a crossover point at $s \approx -0.67$ where the two designs have the same overall preference of 0.62. As $s \to -\infty$ the non-compensating *min* is recovered. The limits $s = 10$ and $s = -10$ were chosen to show the behavior near the crossover point; values of s outside this range are certainly permissible.

Note that the results for the traditional weighted sum fall in the region $s > 0$. Indeed, the weighted sum is identical to $\mathcal{P}_1$.

Conclusion

The Method of Imprecision is a formal system, based on the mathematics of fuzzy sets, for the representation and manipulation of imprecise design information through the specification of preferences on design and performance variables. In practice, designers informally combine imprecise design information using trade-off strategies that depend on the particular design problem. The MoI uses aggregation functions to formally model these different trade-off strategies. The class of functions appropriate for the aggregation of these preferences has been explored in this paper. A parameterized family of functions has been detailed. It was shown that this family spans two continua of possible design strategies, one between the non-compensating *min* and the partially compensating $\mathcal{P}_\Pi$, the two original aggregation functions of the MoI, and one between $\mathcal{P}_\Pi$ and the completely compensating *max*.

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Appendix

Proofs

Proofs of claims made in the text are given here for functions of two arguments; the same proofs hold for functions of several variables. The proofs assume that preferences and weights are all non-zero.

Claim 1 *$\mathcal{P}_0$ is identical to the weighted product of powers $\mathcal{P}_\Pi$:*

$$\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \mathcal{P}_\Pi((\mu_1,\omega_1),(\mu_2,\omega_2))$$

Proof

$$\begin{aligned}
&\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) \\
&\quad= \lim_{s\to 0}\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}} \\
&\quad= \exp^{\lim_{s\to 0}\ln\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}}} \\
&\quad= \exp^{\lim_{s\to 0}\frac{1}{s}\ln\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)} \qquad (1)
\end{aligned}$$

Note that, by the definition of the derivative,

$$\begin{aligned}
&\lim_{s\to 0}\frac{1}{s}\ln\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right) \\
&\quad= \frac{d}{ds}\left[\ln\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)\right]_{s=0} \\
&\quad= \frac{\omega_1\ln{\mu_1}^s+\omega_2\ln{\mu_2}^s}{\omega_1+\omega_2}
\end{aligned}$$

Thus, proceeding from 1, it follows that

$$\begin{aligned}
&\lim_{s\to 0} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) \\
&\quad= \exp^{\frac{\omega_1\ln{\mu_1}^s+\omega_2\ln{\mu_2}^s}{\omega_1+\omega_2}} \\
&\quad= \left({\mu_1}^{\omega_1}{\mu_2}^{\omega_2}\right)^{\frac{1}{\omega_1+\omega_2}} \\
&\quad= \mathcal{P}_\Pi((\mu_1,\omega_1),(\mu_2,\omega_2))
\end{aligned}$$

which proves the claim.

Claim 2 *As $s\to-\infty$, $\mathcal{P}_s$ tends to min:*

$$\lim_{s\to-\infty} \mathcal{P}_s((\mu_1,\omega_1),(\mu_2,\omega_2)) = \min(\mu_1,\mu_2)$$

Proof

$$\begin{aligned}
&\lim_{s\to-\infty}\left(\frac{\omega_1{\mu_1}^s+\omega_2{\mu_2}^s}{\omega_1+\omega_2}\right)^{\frac{1}{s}} \\
&\quad= \lim_{s\to-\infty}\left(\frac{\omega_1}{\omega_1+\omega_2}\right)^{\frac{1}{s}}\left({\mu_1}^s+\frac{\omega_2}{\omega_1}{\mu_2}^s\right)^{\frac{1}{s}} \\
&\quad= \lim_{t\to+\infty}\left(\frac{1}{{\mu_1}^t}+\frac{\omega_2}{\omega_1{\mu_2}^t}\right)^{\frac{1}{-t}} \\
&\quad= \lim_{t\to+\infty}\left(\frac{{\mu_2}^t}{\left(\frac{\mu_2}{\mu_1}\right)^t+\frac{\omega_2}{\omega_1}}\right)^{\frac{1}{t}} \\
&\quad= \lim_{t\to+\infty}\frac{\mu_2}{\left(\left(\frac{\mu_2}{\mu_1}\right)^t+\frac{\omega_2}{\omega_1}\right)^{\frac{1}{t}}}
\end{aligned}$$

Note that if $\mu_2 \le \mu_1$, then the denominator tends to 1 and the limit is μ_2; if $\mu_1 < \mu_1$, then the denominator tends to $\frac{\mu_2}{\mu_1}$ and the limit is μ_1. This proves the claim.

Tables

Table 2 gives the axioms of the MoI. Table 3 lists the properties of the weighted mean.

$\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n)) \le \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n',\omega_n))$ for $\mu_n \le \mu_n'$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n)) \le \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n'))$ for $\omega_n \le \omega_n'$; $\mu_i < \mu_n \;\; \forall i < n$	(monotonicity)
$\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_i,\omega_i),\cdots,(\mu_j,\omega_j),\cdots,(\mu_n,\omega_n)) =$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_j,\omega_j),\cdots,(\mu_i,\omega_i),\cdots,(\mu_n,\omega_n))\; \forall i,j$	(commutativity)
$\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega_k),\cdots,(\mu_n,\omega_n)) =$ $\lim_{\mu_k'\to\mu_k} \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k',\omega_k),\cdots,(\mu_n,\omega_n)) \quad \forall k$ $\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega_k),\cdots,(\mu_n,\omega_n)) =$ $\lim_{\omega_k'\to\omega_k} \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,\omega_k'),\cdots,(\mu_n,\omega_n)) \quad \forall k$	(continuity)
$\mathcal{P}((\mu,\omega_1),\cdots,(\mu,\omega_n)) = \mu$ for $\omega_1,\cdots,\omega_n \ge 0$; $\omega_1+\cdots+\omega_n > 0$	(idempotency)
$\mathcal{P}((\mu_1,\omega_1),\cdots,(0,\omega),\cdots,(\mu_n,\omega_n) = 0$ for $\omega \ne 0$	(annihilation)
$\mathcal{P}((\mu_1,\omega_1 t),\cdots,(\mu_n,\omega_n t)) = \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_n,\omega_n))$ $\forall\, \omega_1,\cdots,\omega_n \ge 0$; $\omega_1+\cdots+\omega_n, t > 0$	(self-scaling weights)
$\mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_k,0),\cdots,(\mu_n,\omega_n))$ $= \mathcal{P}((\mu_1,\omega_1),\cdots,(\mu_{k-1},\omega_{k-1});(\mu_{k+1},\omega_{k+1}),\cdots,(\mu_n,\omega_n))$	(zero weights)

Table 2. Axioms of the MoI (Multiple Arguments).

$\mathcal{P}((\mu,\omega_1),(\mu,\omega_2)) = \mu \quad \forall\, \mu,\omega_1,\omega_2$	(idempotency)
$\exists\, \mu_a < \mu_b$ such that $\forall\, \omega_1,\omega_2 > 0$ $\mu_a = \mathcal{P}((\mu_a,1),(\mu_b,0)) < \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2)) < \mathcal{P}((\mu_a,0)(\mu_b,1)) = \mu_b$	(internality)
$\mathcal{P}((\mu_a,\omega_1 t),(\mu_b,\omega_2 t)) = \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2))$ $\forall\, \omega_1,\omega_2 \ge 0$; $\omega_1+\omega_2, t > 0$	(homogeneity of weights)
$\mathcal{P}[(\mathcal{P}[(\mu_1,\omega_1),(\mu_2,\omega_2)],\omega_1+\omega_2),(\mathcal{P}[(\mu_3,\omega_3),(\mu_4,\omega_4)],\omega_3+\omega_4)]$ $= \mathcal{P}[(\mathcal{P}[(\mu_1,\omega_1),(\mu_3,\omega_3)],\omega_1+\omega_3),(\mathcal{P}[(\mu_2,\omega_2),(\mu_4,\omega_4)],\omega_2+\omega_4)]$	(bisymmetry)
$\mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_2)) < \mathcal{P}((\mu_a,\omega_1),(\mu_b,\omega_3))$ for $\omega_2 < \omega_3 \quad (\mu_a < \mu_b)$	(increasing in weights)
$\mathcal{P}((\mu_1,\omega_1),(\mu_2,\omega_2)) < \mathcal{P}((\mu_1,\omega_1),(\mu_3,\omega_2))$ for $\mu_2 < \mu_3, \quad \omega_2 \ne 0$	(increasing in variables)

Table 3. Properties of the Weighted Mean (Two Arguments).